Higher-Order Computability
5. Exercise Sheet

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Homework

Key to exercises: (P) = programming component (+) = more difficult or open ended.

Exercise H1
Let $F \in C_2$ be defined by $F(\beta) := \beta(0) + \beta(1) + \ldots + \beta(\beta(0))$. Describe two different associates α and α' which both represent F.

Exercise H2
Define the limit space $(\mathbb{N}, \to_0)$ in the usual way by $p_n \to_0 p$ iff there exists some $N \in \mathbb{N}$ such that $\forall n > N(p_n = p)$.

(a) Prove that for any limit space $(X, \to_X)$ we have $\mathscr{C}(\mathbb{N}, X) \cong X^{\mathbb{N}}$.

(b) Prove that for any $\Phi \in \mathscr{C}(\mathscr{C}(\mathbb{N}, X), \mathbb{N})$ and $\alpha \in \mathscr{C}(\mathbb{N}, X) \cong X^{\mathbb{N}}$ there exists some $n \in \mathbb{N}$ such that for any $\beta \in X^{\mathbb{N}}$:

$$\alpha =_n \beta \Rightarrow \Phi(\alpha) = \Phi(\beta).$$

Exercise H3
Prove that the functional $\theta : ((N \to \rho) \to N) \to (N \to \rho) \to \rho^* \to N$ given by

$$\theta_{\omega,\alpha}(s) := \begin{cases} 0 & \text{if } \exists t \preceq s(\omega(\hat{t}) < |t|) \\ 1 + \theta_{\omega,\alpha}(s * \alpha(|s|)) & \text{otherwise} \end{cases}$$

is definable using the scheme of Spector bar recursion

$$\mathrm{SBR}_{\rho,N}(\omega, g, h, s^{\rho^*}) =_N \begin{cases} g(s) & \text{if } \omega(\hat{s}) < |s| \\ h(s, \lambda x.\mathrm{SBR}_{\rho,N}(\omega, g, h, s * x) & \text{otherwise} \end{cases}$$

for suitable g and h.

Exercise H4 (P)
Implement bar recursion as a functional program in your favourite language and run it on some simple input parameters.

Exercise H5 (+)
The scheme of finite bar recursion of type ρ, τ is given by the defining equation:

$$\mathrm{FBR}_{\rho,\tau}(g, h, s) =_\tau \begin{cases} g(s) & \text{if } n < |s| \\ h(s, \lambda x.\mathrm{FBR}_{\rho,\tau}(g, h, s * x)) & \text{if } n \geq |s| \end{cases}$$

Explore the connection between finite bar recursion and primitive recursion in all finite types.