Higher-Order Computability
3. Exercise Sheet

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Homework

Key to exercises: (P) = programming component (+) = more difficult or open ended.

Exercise H1
The drinkers paradox (with a number parameter) is given by

$$\mathrm{DP}\ :\ \forall n \exists x (P(n,x) \Rightarrow \forall y P(n,y))$$

where $P(n,x)$ some decidable predicate in the language of Peano arithmetic.

(a) Argue that in general there is no computable function $f : \mathbb{N} \to \mathbb{N}$ which witnesses the functional interpretation of the drinkers paradox i.e.

$$\forall n, y(P(n, fn) \Rightarrow P(n,y)).$$

(b) On the other hand, find a computable functional $\Phi : \mathbb{N} \times \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ that witnesses the negative translation + functional interpretation of the drinkers paradox i.e.

$$\forall n, g(P(n, \Phi n g) \Rightarrow P(n, g(\Phi n g))).$$

(c) Give a formal description of Φ as both

- a term of System T of type $N \to (N \to N) \to N$,
- an oracle Turing machine.

Exercise H2 (+)
Write down the functional interpretation of the following formula

$$(*) \quad \forall x (\exists y \forall z P(x,y,z) \Rightarrow \exists v Q(x,v)).$$

Explain how, given witnesses for the functional interpretation of $(*)$ together with a witness Φ for the negative translation + functional interpretation of $\forall x \exists y \forall z P(x,y,z)$, we can construct a function $h : \mathbb{N} \to \mathbb{N}$ satisfying $\forall x Q(x, hx)$.

Exercise H3 (+)
Repeat exercise H1, but this time with the following *minimum principle*:

$$\exists i A(i) \Rightarrow \exists j (A(j) \wedge \forall k < j \neg A(k)).$$

where $A(i) :\equiv \exists x Q(i,x)$ for some decidable formula $Q(i,x)$.