Higher-Order Computability
2. Exercise Sheet

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Homework

Key to exercises: (P) = programming component (+) = more difficult or open ended.

Exercise H1
Suppose that the functional interpretation of A is $\exists x \forall y A_D(x,y)$. What is the functional interpretation of $\neg A$, where $\neg A$ is shorthand for $(A \Rightarrow 0 = 1)$? What about $\neg\neg A$?

Exercise H2
Define terms which witness the functional interpretation of $A \Rightarrow \neg\neg A$ (you may assume that $P \Leftrightarrow \neg\neg P$ for all quantifier free formulas of System T).

Exercise H3
What is the functional interpretation of the axiom of contraction $A \Rightarrow A \wedge A$? Can you give concrete witnessing terms?

Hint. For the second part, you should make use of the following fact: For any formula P of System T one can construct a characteristic term $t_P : N$ whose free variable are the same as those of P and which satisfies $(t_P = 0 \Leftrightarrow P)$.

Exercise H4 (+)
In the lectures, we studied the following theorem of Heyting arithmetic

$$\forall n \exists m (m > n \wedge Q(m)) \Rightarrow \forall k \exists x (x = [x_1, \ldots, x_k] \wedge x_1 < \ldots < x_k \wedge (\forall i \leq k) Q(x_i))$$

where Q is quantifier-free, and by $x = [x_1, \ldots, x_k]$ we simply mean that x codes the finite sequence $[x_1, \ldots, x_k]$. A partial functional interpretation of this is given by

$$\exists \Phi \forall f, k (\forall n (f\,n > n \wedge Q(f\,n)) \Rightarrow \Phi(f,k) = [x_1, \ldots, x_k] \wedge x_1 < \ldots < x_k \wedge (\forall i \leq k) Q(x_i))$$

which is witnessed by $\Phi(f,k) := [f(0), f^{(2)}(0), \ldots, f^{(k)}(0)]$. The complete functional interpretation also requires a functional Ψ witnessing the universal quantifier $\forall n$ in the *premise* of the implication:

$$\exists \Phi, \Psi \forall f, k (f(\Psi(f,k)) > \Psi(f,k) \wedge Q(f(\Psi(f,k)))) \Rightarrow \Phi(f,k) = [x_1, \ldots, x_k] \wedge x_1 < \ldots < x_k \wedge (\forall i \leq k) Q(x_i)).$$

Can you find a concrete functional Ψ which works?

Hint. Your solution to Exercise H3 may help.

Exercise H5 (P+)
Write a program which takes as input an arbitrary formula in the language of Heyting arithmetic and returns its functional interpretation.

Exercise H6 (+)
Do some research on the internet for some recent papers on the extraction of programs from proofs. What techniques are used aside from the Dialectica interpretation? What are the main research themes being explored at the moment? Are there any specific case studies you find interesting?