How Deep is the Dialectica?

Thomas Powell
University of Bath

(jww Anupam Das)

StrIP Kick-Off Workshop

University of Birmingham
9 June 2022

These slides will be available at
`https://t-powell.github.io/talks`

Gödel's Dialectica

Conceived by Gödel in the 1930's, published in his landmark 1958 paper [1].

Maps formulas A to an equivalent formula of the form $A^D := \exists x \forall y A_D(x, y)$, where $A_D(x, y)$ is quantifier-free:

$$
\begin{aligned}
A^D &:= A \quad \text{if } A \text{ is computationally neutral} \\
(A \wedge B)^D &:= \exists x, u \forall y, v (A_D(x, y) \wedge B_D(u, v)) \\
(A \vee B)^D &:= \exists b, x, u \forall y, v (A_D(x, y) \vee_b B_D(u, v)) \\
(A \Rightarrow B)^D &:= \exists f, g \forall x, v (A_D(x, gxv) \Rightarrow B_D(fx, v)) \\
(\exists z\, A)^D &:= \exists z, x \forall u\, A_D(x, u) \\
(\forall z\, A)^D &:= \exists f \forall z, u\, A_D(fz, y)
\end{aligned}
$$

Theorem. *If* HA $\vdash A$*, then there is a term t of System T such that* System T $\vdash \forall y\, A_D(t, y)$.
If PA $\vdash A$*, then there is a term t of System T such that* System T $\vdash \forall y\, (A^{\neg\neg})_D(t, y)$.

---

[1] Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes (On a hitherto unexploited extension of the finitary standpoint), **dialectica**, 1958

The significance of the Dialectica

1. Foundational achievements:
   - Relative consistency proofs for arithmetic and later analysis.
   - Characterisation of provably recursive functions.
   - Expansion of interpretation to a wide range of logical systems.
   - Recent use by Normann-Sanders in higher-order reverse mathematics and computability theory.
2. Program extraction and "proof mining":
   - Can be used to extract numerical information from seemingly nonconstructive proofs.
   - Monotone variants can be used to eliminate compactness arguments.
   - Development of specialist soundness proofs that apply to theories for reasoning about abstract spaces.
3. Categories, types and games:
   - Inspired the Dialectica categories, which form one of the first models of linear logic.
   - Connection between Dialectica and learning/sequential games.
   - Recent stuff involving state monad or abstract machines or differential lambda calculus...

An open question

The Dialectica interpretation:

- is amazingly subtle,
- has a phenomenal track record of useful applications,
- always surprises you.

Why not throw it at deep inference and see what happens?

More seriously:

- Classical deep inference and the classical Dialectica possess symmetry. Do these align, or can they be made to align?
- The Dialectica has been applied to many different logics (arithmetic, analysis, predicate logic, linear logic, nonstandard analysis, ...), but the impact of different proof systems has not really been explored.
- The set of people who study Dialectica is almost completely disjoint from the set of people who know what deep inference is. What would happen if they talk to each other?

A starting point

The Dialectica was adapted to first-order predicate logic by Gerhardy and Kohlenbach[2].

They use a handy combination of $\neg\neg$ + Dialectica called the Shoenfield interpretation. It's necessary to introduce case distinction constants χ_A for all quantifier-free formulas A.

Example (Drinkers' paradox)

We have $\mathrm{PL} \vdash \exists x\,(P(x) \to \forall y\, P(x))$.

The Shoenfield interpretation requires us to witness $\forall f \exists x\,(P(x) \to P(fx))$.

Assuming we have at least one constant c, this is done by

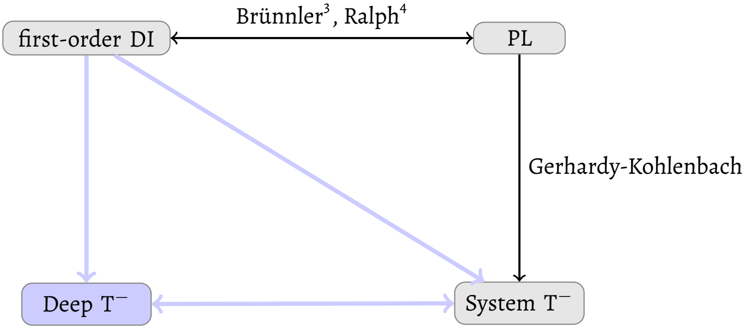
$$\Phi(f) := \chi_{P(fc)}(c)(fc) \quad \text{i.e.} \quad \Phi(f) = \begin{cases} c & \text{if } P(fc) \\ fc & \text{if } \neg P(fc) \end{cases}$$

Looking at all possible instantiations of $\chi_{P(fc)}$ gives us an Herbrand disjunction for the drinkers paradox:

$$(P(c) \to P(fc)) \vee (P(fc) \to P(f(fc)))$$

[2] Extracting Herbrand disjunctions by functional interpretation, **Arch. Math. Logic**, 2005.

A design problem



[3] Deep Inference and Symmetry in Classical Proofs, **PhD thesis**, 2003

[4] Modular Normalisation of Classical Proofs, **PhD thesis**, 2019

Some initial optimism

There are cases where quantifier symmetry of the Dialectica matches the up/down symmetry of deep inference:

$$
\begin{aligned}
&[\forall x \exists y\, P(x,y) \Rightarrow \exists u \forall v\, Q(u,v)]^{\neg\neg} \\
\rightsquigarrow\, &\forall x \exists y\, P(x,y) \Rightarrow \neg\neg \exists u \forall v\, Q(u,v) \\
\rightsquigarrow\, &\exists f \forall x\, P(x,fx) \Rightarrow \forall g \exists u\, Q(u,gu) \\
\rightsquigarrow\, &\forall f, g \exists x, u [P(x,fx) \Rightarrow Q(u,gu)]
\end{aligned}
$$

$$
\begin{aligned}
&[\overline{\exists u \forall v\, Q(u,v)} \Rightarrow \overline{\forall x \exists y\, P(x,y)}]^{\neg\neg} \\
=\, &[\forall u \exists v\, \bar{Q}(u,v) \Rightarrow \exists x \forall y\, \bar{P}(x,y)]^{\neg\neg} \\
\rightsquigarrow\, &\forall u \exists v\, \bar{Q}(u,v) \Rightarrow \neg\neg \exists x \forall y\, \bar{P}(x,y) \\
\rightsquigarrow\, &\exists g \forall u\, \bar{Q}(u,gu) \Rightarrow \forall f \exists x\, \bar{P}(x,fx) \\
\rightsquigarrow\, &\forall f, g \exists x, u [\bar{Q}(u,gu) \Rightarrow \bar{P}(x,fx)]
\end{aligned}
$$

Alas the symmetry breaks down in general, but we do get some partial symmetry.

So what happens when you throw the Dialectica at deep inference?

A derivation $\pi \, \begin{array}{c} A \\ \| \\ B \end{array}$ in deep inference corresponds to the classical formula $\neg A \vee B$.

The Schoenfield (= $\neg\neg$ + Dialectica) interpretation of the latter is

$$\forall f, u \exists x, v(\neg A_S(x, fx) \vee B_s(u, v))$$

This would be interpreted by a pair of terms X and V in arguments f, u i.e.

$$\forall f, u(\neg A_S(X_{f,u}, f(X_{f,u})) \vee B_S(u, V_{f,u}))$$

Thought experiment

We imagine that we have already set up Deep T, a quantifier-free system of deep inference that can deal with higher order lambda terms with case distinctions.

Instead of a proof of the above disjunction we would aim for the following derivation in Deep T:

$$\begin{array}{c} A_S(X_{f,u}, f(X_{f,u})) \\ \| \\ B_S(u, V_{f,u}) \end{array}$$

Try it and see what comes out[5]

Operations (ignore the term syntax, it's just there to illustrate a point):

$$
\begin{aligned}
(\pi \star \pi')_S &:= \pi_S[f \Leftarrow \lambda x.\alpha x W_{\beta x,p}] \star \pi'_S[g \Leftarrow \beta \tilde{X}_{\alpha,\beta,u,p}] \ \text{ for } \star \in [\vee, \wedge] \\
(\exists z \pi)_S &:= \pi_S[z, u, f \Leftarrow \zeta, \Psi\zeta G, F\Phi] \\
(\forall z \pi)_S &:= \text{similar to but simpler than existential} \\
(\pi \circ \pi')_S &:= \pi_S[u \Leftarrow U_{p,g_f}] \circ \pi'_S[g \Leftarrow g_f]
\end{aligned}
$$

Inference rules:

equality rules	simple substitutions, then reduces to itself
switch and medial	same as above
atomic structural rules	trivial: no computational content
quantifier equality rules	simple substitution
quantifier rules (easy ones)	simple subsitution
quantifier rules (hard ones)	big case distinctions

[5] We took SKSq (atomic first-order system with cut)

A very rough illustration

Example (Drinkers' paradox)

$$
\cfrac{\cfrac{\cfrac{\mathtt{t}}{\forall x\left\{\cfrac{\mathtt{t}}{P(x)\vee\bar{P}(x)}\,ai\!\downarrow\right\}}\;=}{\exists x\,\bar{P}(x)\vee\forall y\,P(y)}\;u\!\downarrow}{\exists x(\bar{P}(x)\vee\forall y\,P(y))}\;r2\!\uparrow
$$

The above gets translated to something like the following in Deep T^{-}:

$$
\cfrac{\cfrac{\mathtt{t}}{\left\{\cfrac{P(fc)}{\cfrac{\mathtt{f}}{\bar{P}(c)}\,aw\!\downarrow\;\vee P(fc)}\;=\right\}\vee\left\{\cfrac{\bar{P}(fc)}{\bar{P}(fc)\vee\cfrac{\mathtt{f}}{P(f(fc))}\,aw\!\downarrow}\;=\right\}}\;ai\!\downarrow}{\bar{P}(\Phi f)\vee P(f(\Phi f))}\;\chi
$$

for $\Phi(f) := \chi_{P(fc)}(c)(fc)$.

Potential benefits future

- A new proof of Herbrand's theorem via deep inference.
- New deep inference style calculi for reasoning about higher-order logic and programs.
- Insight into the computational behaviour of deep inference proofs. Which rules and operations are computationally complex?
- Insight into structural properties of the Dialectica. Can we use DI to explain symmetry in the Dialectica?
- A change in emphasis in the way we view the Dialectica, more in line with the ethos of deep inference: We are interesting in extracting natural, expressive programs rather than just proving that they exist!

THANK YOU!