A proof theoretic analysis of Littlewood's Tauberian theorems

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These slides are available at
`https://t-powell.github.io/talks`

Content of this talk

1. A six slide introduction to applied proof theory!
2. An overview of Tauberian theory.
3. A Dialectica interpretation of Littlewood's theorem.
4. Some new remainder estimates.

Applied Proof Theory

What is applied proof theory?

There is a famous quote due to G. Kreisel (*A Survey of Proof Theory II*):

> *"What more do we know when we know that a theorem can be proved by limited means than if we merely know that it is true?"*

In other words, the **proof** of a theorem gives us much more information than the mere **truth** of that theorem.

Applied proof theory is a branch of logic that uses proof theoretic techniques to exploit this phenomenon.

Everyone does applied proof theory

Problem. Give me an upper bound on the nth prime number p_n.

1. What is p_n? I know it exists because of Euclid...
2. Specifically, given $p_1, \ldots, p_{n-1}$, I know that $N := p_1 \cdot \ldots \cdot p_{n-1} + 1$ contains a *new* prime factor q, and so $p_n \leq q \leq N$.
3. In other words, the sequence $\{p_n\}$ satisfies

$$p_n \leq p_1 \cdot \ldots \cdot p_{n-1} + 1 \leq (p_{n-1})^{n-1}$$

4. By induction, it follows that e.g. $p_n < 2^{2^n}$.

This is a simple example of applied proof theory in action! From the **proof** that there are infinitely many primes, we have inferred a **bound** on the nth prime.

… but it's not always that simple

Theorem (Littlewood 1914)

The functions of integers

(a) $\psi(x) - x$, *and*

(b) $\pi(x) - li(x)$

change signs infinitely often, where $\pi(x)$ *is the number of prime* $\leq x$, $\psi(x)$ *is the is logarithm of the l.c.m. of numbers* $\leq x$ *and* $li(x) = \int_0^x (1/\log(u))du$.

The original proof is utterly nonconstructive, using among other things a **case distinction on the Riemann hypothesis**. At the time, no numerical value of x for which $\pi(x) > li(x)$ was known.

In 1952, Kreisel analysed this proof and extracted recursive bounds for sign changes (*On the interpretation of non-finitist proofs, Part II*):

> *"Concerning the bound … note that it is to be expected from our principle, since if the conclusion … holds when the Riemann hypothesis is true, it should also hold when the Riemann hypothesis is nearly true: not all zeros need lie on* $\sigma = \frac{1}{2}$*, but only those whose imaginary part lies below a certain bound … and they need not lie on the line* $\sigma = \frac{1}{2}$*, but near it"*

What applied proof theory looks like today

Theorem (Kirk and Sims, *Bulletin of the Polish Academy of Sciences* 1999)

Suppose that C is a closed subset of a uniformly convex Banach space and $T : C \to C$ is asymptotically nonexpansive with $\mathrm{int}(\mathrm{fix}(T)) \neq \emptyset$. Then for each $x \in C$ the sequence $\{T^n x\}$ converges to a fixed point of T.

Theorem (P., *Journal of Mathematical Analysis and Applications* 2019)

Let $T : C \to C$ be a nonexpansive mapping in L_p for $2 \leq p < \infty$, and suppose that $B_r[q] \subset \mathrm{fix}(T)$ for some $q \in L_p$ and $r > 0$. Suppose that $x \in C$ and $\|x - q\| < K$, and define $x_n := T^n x$. Then for any $\varepsilon > 0$ we have

$$\forall n \geq f(\varepsilon)(\|Tx_n - x_n\| \leq \varepsilon)$$

where

$$f(\varepsilon) := \left\lceil \frac{p \cdot 2^{3p+1} \cdot K^{p+2}}{\varepsilon^p \cdot r^2} \right\rceil$$

How does proof theory come in to play?

We obtained a bound on the *n*th prime from Euclid's proof without any special techniques. However, serious applications usually involve some of the following, either implicitly or explicitly:

- proof interpretations, particularly Gödel's Dialectica,
- computability and complexity in higher types,
- logical relations (particularly *majorizability*),
- formal systems and type theory.

Typically, one also needs to do some serious *mathematics* as well!

What makes an area of mathematics attractive for proof mining?

1. *Proofs* are non-trivial, and use subtle nonconstructive lemmas, but *theorems* are 'nice' from a proof theoretic perspective.

2. Numerical information is relevant in that area.

3. There are many variations of key proof tactics in different settings. Proof theoretic insights can then lead to qualitative results, generalisations and unification.

Tauberian theory

Abel's theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$\sum_{i=0}^{\infty} a_i = s$$

it follows that

$$F(x) \to s \quad \text{as} \quad x \nearrow 1.$$

This is a classical result in elementary analysis called **Abel's theorem** (N.b. it also holds in the complex setting). You can use it to e.g. prove that

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} = \log(2).$$

Does the converse of Abel's theorem hold?

NO.

For a counterexample, define $F : (-1, 1) \to \mathbb{R}$ by

$$F(x) = \frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i x^i$$

Then

$$F(x) \to \frac{1}{2} \quad \text{as} \quad x \nearrow 1$$

but

$$\sum_{i=0}^{\infty} (-1)^i \quad \text{does not converge}$$

Tauber's theorem fixes this

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$F(x) \to s \quad \text{as} \quad x \nearrow 1 \quad \text{AND} \quad |na_n| \to 0$$

it follows that

$$\sum_{i=0}^{\infty} a_i = s$$

This is **Tauber's theorem**, proven in 1897 by Austrian mathematician Alfred Tauber (1866 - 1942).

Tauberian theorems

The basic structure of Tauber's theorem is:

$$\text{Let} \quad F(x) = \sum_{i=0}^{\infty} a_i x^i$$

Then if we know

(A) Something about the behaviour of $F(x)$ as $x \nearrow 1$

(B) Something about the growth of $\{a_n\}$ as $n \to \infty$

Then we can conclude

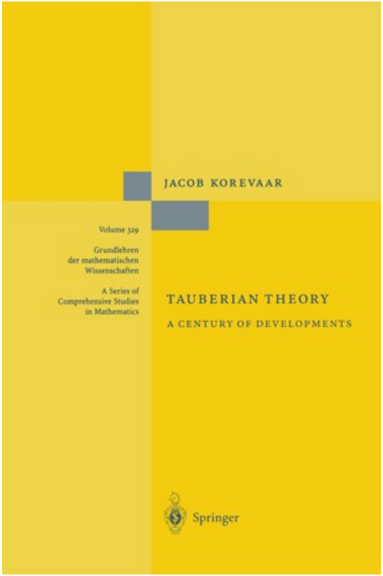
(C) Something about the convergence of $\sum_{i=0}^{\infty} a_i$.

This basic idea has been **considerably generalised** e.g. for

$$F(s) := \int_1^{\infty} a(t) t^{-s} \, \mathrm{d}t$$

and has grown into an area of research known as **Tauberian Theory**.

There is now a whole textbook (published 2004, 501 pages)



Tauber's theorem was first extended by Littlewood (1911)

THE CONVERSE OF ABEL'S THEOREM ON POWER SERIES

By J. E. Littlewood.

[Received September 28th, 1910.—Read November 10th, 1910.—
Revised December, 1910.*]

Introduction.

Abel's theorem states that if $\sum_0^\infty a_n$ is convergent, then $\lim \sum_0^\infty a_n x^n$ exists as $x \to 1$ by real values, and is equal to Σa_n. The converse theorem, however, that the existence of $\lim_{x\to 1} \Sigma a_n x^n$ implies the convergence of Σa_n, is very far from being true; for example, either the Cesàro or the Borel summability of Σa_n suffices for the existence of Abel's limit. It is known, however, that the existence of this limit, *combined with certain conditions satisfied by the a's*, does imply the convergence of Σa_n. Three such sets of conditions, for example, are:

(*a*)† the a's are all positive;

(*b*) the order of a_n has a certain upper limit;

(*c*)‡ the function $\Sigma a_n x^n$ is regular at the point $x = 1$ and $a_n \to 0$.

In the present paper we are concerned with the problems arising out of case (*b*), where the only additional restriction on the a's is an upper limit to the order of a_n. The theorem of this case is due to M. Tauber.§ The result is remarkable and apparently paradoxical in view of Abel's theorem, for it may be expressed roughly by saying that if Σa_n is not

Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$F(x) \to s \quad \text{as} \quad x \nearrow 1 \quad \text{AND} \quad |na_n| \leq C$$

for some constant C, it follows that

$$\sum_{i=0}^{\infty} a_i = s$$

One of Littlewood's first major results. In *A Mathematical Education* he writes (of this period)

> *" On looking back this time seems to me to mark my arrival at a reasonably assured judgement and taste, the end of my "education". I soon began my 35-year collaboration with Hardy."*

One of first papers of this collaboration (1914):

TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

By G. H. Hardy *and* J. E. Littlewood.

[Received October 3rd, 1913.—Read November 13th, 1913.]

1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let
$$f(x) = \Sigma a_n x^n$$
be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let
$$s_n = a_0 + a_1 + \ldots + a_n,$$
$$L(u) = (\log u)^{a_1} (\log \log u)^{a_2} \ldots,$$
where the a's are real. Then it is known that, *if*
$$s_n \sim A n^a L(n),$$
where $A \neq 0$, *as* $n \to \infty$, *the indices* a, a_1, a_2, ... *being such that* $n^a L(n)$ *tends to a positive limit or to infinity, then*

The Hardy-Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that $\sum_{i=0}^{\infty} a_i x^i$ converges for $|x| < 1$. Then whenever

$$(1 - x) \sum_{i=0}^{\infty} a_i x^i \to s \quad \text{as} \quad x \nearrow 1 \;\; \text{AND} \;\; a_n \geq -C$$

for some constant C, it follows that

$$\frac{1}{n} \sum_{i=0}^{n} a_i \to s \quad \text{as} \quad n \to \infty$$

They later used this result to give a new proof of the *prime number theorem*:

$$\pi(x) \sim \frac{x}{\log(x)}$$

Rates of convergence have been studied for over 70 years!

MATHEMATICS

BEST L_1 APPROXIMATION AND THE REMAINDER IN LITTLEWOOD'S THEOREM [1])

BY

JACOB KOREVAAR

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of March 28, 1953)

1. *Introduction and results.* Let $f(x)$ be continuous on $a \leqslant x \leqslant b$ and satisfy a LIPSCHITZ condition of order 1:

(1. 1) $$|f(x_1) - f(x_2)| \leqslant A|x_1 - x_2| \text{ for all } x_1,\ x_2 \text{ on } a \leqslant x \leqslant b.$$

D. JACKSON [2] has shown that for such an $f(x)$ there are a constant D and a sequence of polynomials $p_n(x)$ of degree n, $n = 1, 2, \ldots$, such that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| < D/n.$$

In this paper we consider approximation to functions $f(x)$ which are continuous on $a \leqslant x \leqslant b$ except for a finite number of jump discontinuities, and which satisfy a LIPSCHITZ condition (1. 1) on each of the subintervals of $a \leqslant x \leqslant b$ on which they are continuous ("functions of class $J(a, b)$"). It follows from results by NIKOLSKY [7] that for any such function $f(x)$ there still are a constant D_1 and a sequence of polynomials $p_n(x)$ of degree n such that

Remainder estimate for Littlewood's theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that there is some $b > 0$ such that

$$\sum_{i=0}^{\infty} a_i x^i = s + \mathcal{O}\left\{(1-x)^b\right\}$$

as $x \nearrow 1$. Then

$$\sum_{i=0}^{\infty} a_i = s + \mathcal{O}\left(\frac{1}{\log(n)}\right)$$

as $n \to \infty$.

This is essentially a quantitative version of Littlewood's theorem, relating a rate of convergence for the premise to a rate of convergence for the conclusion.

What makes an area of mathematics attractive for proof mining?

1. *Proofs* are non-trivial, and use subtle nonconstructive lemmas, but *theorems* are 'nice' from a proof theoretic perspective.

2. Numerical information is relevant in that area.

3. There are many variations of key proof tactics in different settings. Proof theoretic insights can then lead to qualitative results, generalisations and unification.

What makes Tauberian theory attractive for proof mining?

1. Tauberian theorems have a simple logical structure, but are often extremely deep and difficult to prove. Littlewood's theorem uses results from approximation theory.

2. There is an existing interest in quantitative versions of Tauberian theorems in the form of remainder estimates.

3. There are lots of variations of Tauberian theorems, all following the general structure

$$\text{convergence} + \text{growth condition} \implies \text{convergence}$$

and using similar lemmas in their proofs. Initial success could lead to a promising area of research in proof mining.

A Dialectica interpretation of Littlewood's theorem

Some initial results on quantitative Tauberian theorems given in P. 2020:

A note on the finitization of Abelian and Tauberian theorems

Thomas Powell

Abstract

We present finitary formulations of two well known results concerning infinite series, namely Abel's theorem, which establishes that if a series converges to some limit then its Abel sum converges to the same limit, and Tauber's theorem, which presents a simple condition under which the converse holds. Our approach is inspired by proof theory, and in particular Gödel's functional interpretation, which we use to establish quantitative version of both of these results.

1 Introduction

In an essay of 2007 [17] (later published as part of [18]) T. Tao discussed the so-called *correspondence principle* between 'soft' and 'hard' analysis, whereby many *infinitary* notions from analysis can be given an equivalent *finitary* formulation. An important instance of this phenomenon is provided by the simple concept of Cauchy convergence of a sequence $\{c_n\}$:

$$\forall \varepsilon > 0 \; \exists N \; \forall m, n \geq N \; (|c_m - c_n| \leq \varepsilon).$$

This corresponds to the finitary notion of $\{c_n\}$ being *metastable*, which is given by the following formula:

$$\forall \varepsilon > 0 \; \forall g : \mathbb{N} \to \mathbb{N} \; \exists N \; \forall m, n \in [N; N + g(N)] \; (|c_m - c_n| \leq \varepsilon), \tag{1}$$

where $[N; N+k] := \{N, N+1, \ldots, N+k-1, N+k\}$. Roughly speaking, a sequence $\{c_n\}$ is metastable if for any given error $\varepsilon > 0$ it contains a finite regions of stability of any 'size', where size is represented by the function $g : \mathbb{N} \to \mathbb{N}$.

The equivalence of Cauchy convergence and metastability is established via purely logical reasoning, and indeed, as was quickly observed, the correspondence principle as presented in [17] has deep connections with proof theory. More specifically, the finitary variant of an infinitary statement is typically closely related to its *classical Dialectica interpretation* [1], which provides a general method for obtaining quantitative versions of mathematical theorems.

Finitary formulations of infinitary properties play a central role in the *proof mining* program developed by U. Kohlenbach from the early 90s [7]. Here, it is often the case that a given mathematical theorem has, in general, no computable realizer (for Cauchy convergence this is demonstrated by the existence of so-called *Specker sequences* [16], which will be discussed further in Section 3). On the other hand, the corresponding finitary formulation can typically not only be realized, but a realizer can be directly extracted from a proof that the original property holds. The extraction of a computable bound $\Omega(\varepsilon, g)$ on N in (1) – a so-called *rate*

A “Cauchy” reformulation of Littlewood’s theorem

For the rest of this talk, $\{a_n\}$ is a sequence of reals and $\{s_n\}, F : [0, 1) \to \mathbb{R}$ are defined by

$$s_n := \sum_{i=0}^{n} a_i \quad F(x) := \sum_{i=0}^{\infty} a_i x^i$$

Theorem (Littlewood’s theorem – Cauchy variant, P. 2022)

Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and that

$$\forall \delta > 0 \exists M \forall x, y \in [e^{-1/M}, 1)(|F(x) - F(y)| \leq \delta)$$

Then we have

$$\forall \varepsilon > 0 \exists N \forall m, n \geq N(|s_n - F(e^{-1/m})| \leq \varepsilon)$$

Proof strategy (Karamata's method)

1. All functions $f : [0, 1] \to \mathbb{R}$ which are continuous aside from a finite number of jump discontinuities, and Lipschitz on all continuous subintervals, can be approximated to arbitrary precision by a polynomial.
2. Let $\chi : [0, \infty) \to \mathbb{R}$ be the characteristic function of $[0, 1]$ and for $\varepsilon > 0$ show that there exists a polynomial P_ε with $P_\varepsilon(0) = 0$ and $P_\varepsilon(1) = 1$ such that

$$\int_0^\infty \frac{|\chi(t) - P_\varepsilon(t)|}{t} dt < \varepsilon$$

3. Let $a : [0, \infty) \to \mathbb{R}$ be defined by $a(t) = a_n$ for $t \in [n, n + 1)$, and let

$$I_{P_\varepsilon}(n) := \int_0^\infty a(t) P_\varepsilon(e^{-t}) dt$$

4. Now carry out two calculations:
   (i) Using $a_n = \int_0^\infty a(t)\chi(t/n)dt$ and $|a_n| \leq C/n$ show that

   $$|s_n - I_{P_\varepsilon}(n)| \leq \varepsilon/2$$

   (ii) Using that P_ε is a polynomial and $|s_n - F(e^{-1/m})| \to 0$ show that for sufficiently large m, n we have

   $$|I_{P_\varepsilon}(n) - F(e^{-1/m})| \leq \varepsilon/2$$

Logical structure of Littlewood's theorem

Given $\{a_n\}$ and $C > 0$, define $A(p)$, $B(\delta, M, l)$ and $D(\varepsilon, N, k)$ by

$$A(p) := \forall n \leq p(n|a_n| \leq C)$$
$$B(\delta, M, l) := \forall x, y \in [e^{-1/M}, e^{-1/(M+l)}](|F(x) - F(y)| \leq \delta)$$
$$D(\varepsilon, N, k) := \forall m, n \in [N, N+k](|s_m - F(e^{-1/m})| \leq \varepsilon)$$

It turns out that we can actually prove the following:

$$\forall \varepsilon [\forall p\, A(p) \wedge \exists M \forall l\, B(\delta_\varepsilon, M, l) \implies \exists N \forall k\, D(\varepsilon, N, k)]$$

where δ_ε is definable from $\varepsilon > 0$ and some information from P_ε.

Now take the Dialectica interpretation of this:

$$\forall \varepsilon, M \exists N \forall k \exists p, l [A(p) \wedge B(\delta_\varepsilon, M, l) \implies D(\varepsilon, N, k)]$$

We should be able to extract witnesses for N, p and l from the proof of Littlewood's theorem.

A result from approximation theory

Definition

We call $\Omega : (0,\infty) \to (0,\infty) \times (0,\infty)$ a modulus of polynomial approximation to χ if for any $\varepsilon > 0$ there exists a polynomial $P_\varepsilon(x) = \sum_{i=1}^{d} c_i x^i$ satisfying $P_\varepsilon(0) = 0$, $P_\varepsilon(1) = 1$ and

$$\int_0^\infty \frac{|\chi(t) - P_\varepsilon(t)|}{t} dt < \varepsilon$$

such that

$$d \leq \Omega_0(\varepsilon) \quad \text{and} \quad \sum_{i=1}^{d} |c_i| \leq \Omega_1(\varepsilon)$$

Lemma

There are constants $A, B > 0$ such that

$$\Omega(\varepsilon) = (\Omega_0(\varepsilon), \Omega_1(\varepsilon)) = \left(\frac{A}{\varepsilon}, B^{1/\varepsilon}\right)$$

is a modulus of polynomial approximation.

Dialectica interpretation of Littlewood's theorem

Theorem (P. 2022)

Let $C > 0$ and suppose that $a > 0$ is a bound on $\{|a_n|\}$. Fix $\varepsilon > 0$ and let

$$(b, v) := \Omega\left(\frac{\varepsilon}{8C}\right) \quad \text{and} \quad \delta := \frac{\varepsilon}{4v}$$

Given some $M \in \mathbb{N}$ define $N \in \mathbb{N}$ by

$$N := b \cdot \max\left\{\left\lceil \frac{L}{\delta} \right\rceil, M\right\} \quad \text{for} \quad L := \frac{a}{1 - e^{-1/M}} + \delta$$

Finally, given $k \in \mathbb{N}$ define $p, l \in \mathbb{N}$ by

$$l := N + k - M \quad \text{and} \quad p := (N + k) \cdot \max\left\{\left\lceil \log\left(\frac{a(N+k)}{\delta}\right)\right\rceil, 1\right\}$$

Then from

$$n|a_n| \leq C \quad \text{for all } n \leq p$$

and

$$|F(x) - F(y)| \leq \delta \quad \text{for all } x, y \in [e^{-1/M}, e^{-1/(M+l)}]$$

it follows that

$$|s_n - F(e^{-1/m})| \leq \varepsilon \quad \text{for all } m, n \in [N, N+k]$$

A game semantics for Littlewood's theorem

∃loise sets out to foil ∀belard's attempt to disprove Littlewood's theorem by showing that $a_n = \mathcal{O}(1/n)$, $F(x) \to s$ and $s_n \not\to s$ all hold together:

1. ∀belard starts by picking some $\varepsilon > 0$, assuming that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and proposing some $M \in \mathbb{N}$ such that $|F(x) - F(y)| \leq \delta$ for all $x, y \in [e^{-1/M}, 1)$. His aim is to show that it is now not the case that $|s_n - F(e^{-1/m})| \leq \varepsilon$ for sufficiently large m, n.
2. ∃loise responds by putting forward an $N \in \mathbb{N}$ such that $|s_n - F(e^{-1/m})| \leq \varepsilon$ for all $m, n \geq N$.
3. ∀belard rejects this by attempting to find a counterexample to the last move, playing $k \in \mathbb{N}$ and claiming that $\varepsilon < |s_n - F(e^{-1/m})|$ for some $n, m \in [N, N+k]$.
4. If ∀belard's attempt worked, then ∃loise responds by producing a pair $l, p \in \mathbb{N}$ which demonstrate that one of ∀belard's original assumptions was false:
    - either $C < n|a_n|$ for some $n \leq p$, or
    - $\delta < |F(x) - F(y)|$ for some $x, y \in [e^{-1/M}, e^{-1/(M+l)}]$.

We have just provided a winning strategy for ∃loise in presenting bounds for winning moves for ∃loise in terms of any play from ∀belard .

Remainder theorems

Remainder theorem 1

Theorem

Suppose that there exists some $C > 0$ *such that* $n|a_n| \leq C$ *for all* $n \in \mathbb{N}$*, and let* $L > 0$ *be a bound on* $|F(x)|$ *for* $x \in [0, 1)$*. Suppose that*

$$\forall \delta > 0 \exists M \leq \phi(\delta) \forall x, y \in [e^{-1/M}, 1)(|F(x) - F(y)| \leq \delta)$$

then we have

$$\forall \varepsilon > 0 \exists N \leq \psi(\varepsilon) \forall m, n \geq N(|s_n - F(e^{-1/m})| \leq \varepsilon)$$

where ψ *is defined by*

$$\psi(\varepsilon) := \Omega_0(\varepsilon/8C) \cdot \max\left\{\left\lceil \frac{L}{\alpha(\varepsilon)} \right\rceil, \phi(\alpha(\varepsilon))\right\} \text{ for } \alpha(\varepsilon) := \frac{\varepsilon}{4\Omega_1(\varepsilon/8C)}$$

Corollary (Exponential rates)

In the special case that

$$\phi(\delta) \leq a\delta^{-b} \text{ for some } a, b > 0$$

substituting in the definition of Ω *and rearranging gives*

$$\psi(\varepsilon) \leq K^{1/\varepsilon} \text{ for a suitable constant } K > 0$$

Recall: Remainder estimate for Littlewood's theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that there is some $b > 0$ such that

$$\sum_{i=0}^{\infty} a_i x^i = s + \mathcal{O}\left\{(1-x)^b\right\}$$

as $x \nearrow 1$. Then

$$\sum_{i=0}^{\infty} a_i = s + \mathcal{O}\left(\frac{1}{\log(n)}\right)$$

as $n \to \infty$.

Rederiving the traditional remainder estimate

If $\sum_{i=0}^{\infty} a_i x^i = s + \mathcal{O}\left\{(1-x)^b\right\}$ then

$$|F(x) - s| \leq a(1-x)^b \quad \text{for some } a > 0$$

which implies that for $x, y \in [e^{-1/M}, 1)$:

$$|F(x) - F(y)| \leq 2a(1 - e^{-1/M})^b \leq 2a/M^b$$

Equivalently, $\phi(\delta) := (2a/\delta)^{-b}$ is a rate of convergence for $|F(x) - F(y)| \to 0$. Thus:

$$\forall m, n \geq K^{2/\varepsilon}(|s_n - F(e^{-1/m})| \leq \varepsilon/2)$$

for suitable K, and therefore applying our Corollary:

$$\forall n \geq K^{2/\varepsilon}(|s_n - s| \leq \varepsilon)$$

Rearranging gives us

$$|s_n - s| \leq 2\log(K)/\log(n)$$

and therefore

$$s_n = s + \mathcal{O}\left(\frac{1}{\log(n)}\right)$$

Remainder theorem 2

We can adapt the standard Specker sequence construction to show that there exist sequences $\{a_n\}$ such that $|F(x) - F(y)| \to 0$ but with no computable rate.

Theorem

Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and let $L > 0$ be a bound on $|F(x)|$ for $x \in [0, 1)$. Suppose that

$$\forall \delta > 0, h : \mathbb{N} \to \mathbb{N} \exists M \leq \Phi(\delta, h) \forall x, y \in [e^{-1/M}, e^{-1/(M+hM)})(|F(x) - F(y)| \leq \delta)$$

then we have

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists N \leq \Psi(\varepsilon, g) \forall m, n \in [N, N + gN](|s_n - F(e^{-1/m})| \leq \varepsilon)$$

where $\Psi(\varepsilon, g) := \beta(\varepsilon, \Phi(\alpha(\varepsilon), h_{\varepsilon,g}))$ for

$$\begin{aligned} h_{\varepsilon,g}(k) &:= \gamma(\alpha(\varepsilon), k, g(\beta(\varepsilon, k)) \\ \alpha(\varepsilon) &:= \frac{\varepsilon}{4\Omega_1(\varepsilon/8C)} \\ \beta(\varepsilon, M) &:= \Omega_0(\varepsilon/8C) \cdot \max\left\{\left\lceil \frac{L}{\alpha(\varepsilon)} \right\rceil, M\right\} \\ \gamma(\varepsilon, M, k) &:= \beta(\varepsilon, M) + k - M \end{aligned}$$

Conclusions

For further details, see:

- P. 2020: **A note on the finitization of Abelian and Tauberian theorems**. Mathematical Logic Quarterly, 66(3): 300–310.
- P. 2022: **A finitization of Littlewood's Tauberian version and an application in Tauberian remainder theory**. Submitted.
- J. Korevaar 2004: **Tauberian Theory: A Century of Developments**. Springer.

Open questions

1. Can we extend these ideas to more complex Tauberian theorems e.g.
   - The Hardy-Littlewood theorem,
   - Integral analogues of Tauberian theorems using Karamata's method,
   - Deeper results involving e.g. Fourier transformations?

2. Are there Tauberian theorems with no known remainder estimates, for which the application of proof-theoretic methods could produce not just generalisations of existing remainder estimates as in this case, but the first ever remainder theorems?

Thank you!