Recent results and open questions in applied proof theory

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These slides will be available at
`https://t-powell.github.io/talks`

What do I do?

Over the last decade, I learned a bit about

$$\text{PROOFS} \mapsto \text{PROGRAMS}$$

Particularly:

- Proof interpretations (like Gödel's Dialectica)
- Higher order computability

These are very deep, I exploit this to do interesting* things in computer science and maths.

*at least to me

What sort of things?

Proof theory in plain view:

- Extending proof interpretations with effects,
- Studying weird forms of recursion and inventing new ones,
- Computational semantics of classical logic

Usually publish this in computer science journals/conferences.

Get lots of ideas from others in the theoretical computer science community.

Undercover work:

- Algorithms in commutative ring theory,
- Convergence results in nonlinear analysis,
- Remainder theorems in Tauberian theory,

Usually publish this in maths or logic journals.

Relatively small number of experts. Progress usually involves reading hundreds of maths papers.

Undercover work (in many cases would be considered ‘proof mining’):

Uses ideas and techniques from proof theory to analyse mathematical proofs and:

- Extract quantitative information (even when the proof is at first glance nonconstructive).
- Obtain generalisations of the original theorem through weakening/abstracting assumptions.
- Give deeper insights into theorems from ‘mainstream’ mathematics and provide a uniform framework through which different results can be brought together.

Common questions/criticisms

- *Who cares about those maths results?*
  - Gödel's Dialectica paper (1958): 997 citations.
  - Definition of weakly contractive mappings (1997): 646 citations.

- *Can you mine theorem X?*

  Possibly, but most likely it won't be useful.

- *Isn't applied proof theory just a sausage machine?*

  No, there is quite some skill involved, firstly in *finding* promising topics, secondly in *applying* the techniques.

- *But I don't see where you used proof theory!*

  It was used but in conjunction with mathematical intuition, so formal details are often omitted.

  If paper was sent to a maths journal, all proof theoretic details usually omitted.

Recent work in nonlinear analysis
(jww Franziskus Wiesnet)

Outline

1. A high level overview
2. A simple worked example
3. A first general result
4. Summary of further results

We start with something familiar:

Throughout this talk, we work in a Banach space X.

A mapping $T : E \to E$ for $E \subseteq X$ is called *strongly contractive* (or often just a *contraction mapping*) if there exists $k \in [0, 1)$ such that $\forall x, y \in E$:

$$\|Tx - Ty\| \leq (1 - k) \|x - y\|$$

Theorem (Banach fixed point theorem)

If T is strongly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \frac{(1-k)^n}{k} \|x_1 - x_0\|$$

$$\boxed{\text{space } X} + \boxed{\text{mapping } T} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence to fixpoint}}$$

A generalisation of the Banach fixed point theorem:

A mapping $T : E \to E$ for $E \subseteq X$ is called ψ-*weakly contractive* if $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

In the case that $\psi(t) := kt$ then T is strongly contractive.

Theorem ([Alber and Guerre-Delabriere, 1997])

If T is weakly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

$\boxed{\text{space } X} + \boxed{\text{mapping } T} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence to fixpoint}}$

Example of a weakly contractive mapping

Define $X = \mathbb{R}$ and $T : [0,1] \to [0,1]$ by $Tx := \sin x$. Then we can show that

$$|\sin x - \sin y| \leq |x-y| - \frac{1}{8}|x-y|^3$$

and so sin is ψ-weakly contractive for $\psi(t) = \frac{1}{8}t^3$.

The unique fixpoint of sin is $x = 0$, and defining $x_{n+1} := \sin x_n$ we have $x_n \to 0$ with rate of convergence

$$x_n \leq \frac{1}{\sqrt{x_0^{-2} + \frac{n-1}{4}}}$$

(cf. [Alber and Guerre-Delabriere, 1997] for details).

A further generalisation:

A mapping $T : E \to E$ for $E \subseteq X$ is called totally asymptotically ψ-weakly contractive if $\psi, \phi : [0, \infty) \to [0, \infty)$ are nondecreasing functions with $\psi(0) = \phi(0) = 0$ and $\psi(t), \phi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

for $k_n, l_n \to 0$. In the case that $k_n = l_n := 0$ then T is ψ-weakly contractive.

Theorem (Adapted from [Alber et al., 2006])

Suppose that $E \subseteq X$ is convex, T is asymptotically ψ-weakly contractive and q is a fixpoint of T. Moreover, from any starting point x_0 define the sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

where $\{\alpha_n\}$ is some sequence of nonnegative reals with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\|x_n - q\|$ is bounded. Then $x_n \to q$.

A clear closed form expression for a rate of convergence is not given in [Alber et al., 2006].

space X	+	mapping T	+	algorithm $\{x_n\}$	$\Longrightarrow$	convergence to fixpoint

First objective: Define a general class of mappings of 'weakly contractive type'

Definition ([Powell and Wiesnet, 2021])

A sequence of mappings $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Example. If T is totally asymptotically ψ-weakly contractive in the sense that

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

then $\{T^n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. any fixpoint of T with modulus

$$\sigma(\delta, c) := \max\left\{f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right)\right\}$$

where f_1, f_2 are rates of convergence for $k_n, l_n \to 0$.

Second objective: Produce general convergence theorems

Theorem (Adapted from [Powell and Wiesnet, 2021])

Suppose that $E \subseteq X$ is convex, $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ. Moreover, from any starting point x_0 define the sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

where $\{\alpha_n\} \in [0, \alpha]$ is some sequence of nonnegative reals with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\|x_n - q\|$ is bounded by $c > 0$. Then $x_n \to q$, with rate of convergence

$$\|x_n - q\| \leq F^{-1}\left(2\Psi(c) - \sum_{i=0}^{n-2} \alpha_i\right)$$

where $F : (0, \infty) \to \mathbb{R}$ is any strictly increasing and continuous function satisfying

$$F(\varepsilon) \geq 2\Psi\left(\frac{\varepsilon}{2}\right) - \alpha \cdot \sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right)$$

and Ψ is given by

$$\Psi(s) := \int^{s} \frac{dt}{\psi(t)}$$

How are these results obtained?

- An analysis of the logical structure of key properties and assumptions.
- An analysis of the convergence proofs (which often use liminfs, convergent subsequences etc).
- A study of the relevant literature, identifying common patterns.

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1 A high level overview

2 A simple worked example

3 A first general result

4 Summary of further results

$T : E \to E$ is ψ-weakly contractive if $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

Theorem (A)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then

$$\|x_{n+1} - q\| \leq \|x_n - q\| - \psi(\|x_n - q\|)$$

for all $n \in \mathbb{N}$.

Proof. We observe that

$$\begin{aligned} \|x_{n+1} - q\| &= \|Tx_n - q\| \quad \text{definition of } x_{n+1} \\ &= \|Tx_n - Tq\| \quad q \text{ a fixpoint of } T \\ &\leq \|x_n - q\| - \psi(\|x_n - q\|) \quad T \text{ is } \psi\text{-weakly contractive} \end{aligned}$$

Lemma (B)

Let $\{\mu_n\}$ be a sequence of nonnegative reals satisfying

$$\mu_{n+1} \leq \mu_n - \psi(\mu_n)$$

where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(t) > 0$ for $t > 0$. Then $\mu_n \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := \left\lceil \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \right\rceil$$

Proof. Suppose for contradiction that there exists $\varepsilon > 0$ such that $\mu_n > \varepsilon$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 1 &\leq \frac{\mu_n - \mu_{n+1}}{\psi(\mu_n)} && \text{(definition of } \mu_n \text{ and } \psi(\mu_n) > 0) \\ &\leq \int_{\mu_{n+1}}^{\mu_n} \frac{dt}{\psi(t)} && (1/\psi(t) \text{ nonincreasing}) \end{aligned}$$

Proof (cont).
For any $N \in \mathbb{N}$ we have

$$\begin{aligned} N &= \sum_{i=0}^{N-1} 1 \\ &\leq \sum_{i=0}^{N-1} \int_{\mu_{n+1}}^{\mu_n} \frac{dt}{\psi(t)} \quad \text{(previous slide)} \\ &\leq \int_{\mu_N}^{\mu_0} \frac{dt}{\psi(t)} \quad (\mu_{n+1} < \mu_n) \\ &\leq \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \quad (\varepsilon < \mu_N) \end{aligned}$$

But this is false for

$$N := \left\lceil \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \right\rceil$$

and therefore there exists some $n \leq N$ such that $\mu_n \leq \varepsilon$. But then in particular, since

$$\mu_{n+1} \leq \mu_n - \psi(\mu_n) \leq \mu_n$$

it follows that $\mu_n \leq \varepsilon$ for all $n \geq N$.

Theorem (= Theorem A + Lemma B)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then $\|x_n - q\| \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := \left\lceil \int_{\varepsilon}^{\|x_0 - q\|} \frac{dt}{\psi(t)} \right\rceil$$

This is a perfectly satisfactory quantiative convergence theorem, where we provide a 'proof theorist's' rate of convergence for $\mu_n \to 0$ i.e. a function Φ such that

$$\forall \varepsilon > 0, \forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

Analysts, on the other hand, typically formulate a rate of convergence as a function f such that

$$\forall n(\mu_n \leq f(n))$$

where $f(n) \to 0$ as $n \to \infty$.

Rate conversion. We have shown that for any $\varepsilon > 0$ we have $\|x_n - q\| \leq \varepsilon$ for

$$n \geq \left\lceil \int_{\varepsilon}^{\|x_0 - q\|} \frac{dt}{\psi(t)} \right\rceil$$

We now want to find for each $n \in \mathbb{N}$ some ε_n such that

$$\|x_n - q\| \leq \varepsilon_n$$

This would work for any ε_n with

$$n - 1 < \int_{\varepsilon_n}^{\|x_0 - q\|} \frac{dt}{\psi(t)} = \Psi(\|x_0 - q\|) - \Psi(\varepsilon_n) \leq n$$

so define ε_n such that

$$\Psi(\|x_0 - q\|) - \Psi(\varepsilon_n) = n$$

i.e.

$$\varepsilon_n := \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

Theorem (= Theorem A + Lemma B + rate conversion)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then $\|x_n - q\| \to 0$, and moreover, for any $n \in \mathbb{N}$ we have

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

Now compare this to:

Theorem ([Alber and Guerre-Delabriere, 1997])

If T is weakly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

General route to a convergence theorem

1. Reduce everything to a recursive inequality in terms of $\mu_n := \|x_n - q\|$.
2. Apply a general quantitative convergence theorem for this inequality.
3. Convert proof-theoretic rate into analyst's rate (optional, but essential if we want to compare with known bounds in simple cases).

Steps 2 and 3 can be done in a very general setting, so that in concrete cases, we only need to adapt Step 1!

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2 A simple worked example

3 A first general result

4 Summary of further results

A sequence $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t. q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Theorem (A^+)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, for any $\delta > 0$ and $n \geq \sigma(\delta, c)$ we have:

$$\|x_{n+1} - q\| \leq \|x_n - q\| - \alpha_n \psi(\|x_n - q\|) + \alpha_n \delta$$

Proof. We observe that for $n \geq \sigma(\delta, c)$

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(A_n x_n - q)\| && \text{(rearranging)} \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n \|A_n x_n - q\| && \text{(triangle inequality)} \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n(\|x_n - q\| - \psi(\|x_n - q\|) + \delta) && \text{(property of } \{A_n\}) \\
&= \|x_n - q\| - \alpha_n \psi(\|x_n - q\|) + \alpha_n \delta
\end{aligned}$$

Lemma (B^+)

Let $\{\mu_n\}$ be a sequence of nonnegative reals such that for any $\delta > 0$ we have

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n \delta$$

for all $n \geq \sigma(\delta)$, where:

- $\psi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function with $\psi(t) > 0$ for $t > 0$;*
- $\{\alpha_n\} \subset [0, \alpha]$ *is a sequence of nonnegative real numbers such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence $r : (0, \infty) \times (0, \infty) \to \mathbb{N}$ i.e.*

$$\forall N \in \mathbb{N}, x > 0 \left(\sum_{n=N}^{r(N,x)} \alpha_n > x \right)$$

Then $\mu_n \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2} \min\left\{ \psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha} \right\} \right), 2 \int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)} \right)$$

and c is an upper bound for $\{\mu_n\}$.

Theorem (= Theorem A$^+$ + Lemma B$^+$)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence r. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, we have $\|x_n - q\| \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right), 2\int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)}\right)$$

Recall from earlier...

Definition ([Powell and Wiesnet, 2021])

A sequence of mappings $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Example. If T is totally asymptotically ψ-weakly contractive in the sense that

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

then $\{T^n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. any fixpoint of T with modulus

$$\sigma(\delta, c) := \max\left\{f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right)\right\}$$

where f_1, f_2 are rates of convergence for $k_n, l_n \to 0$.

Corollary (Quantitative version of [Alber et al., 2006])

Suppose that $T : E \to E$ *is quasi totally asymptotically* ψ*-weakly contractive in the sense that*

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

for $k_n, l_n \to 0$*, that* q *is a fixpoint of* T *and that the sequence* $\{x_n\}$ *satisfies*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for $\{\alpha_n\}$ *a sequence of nonnegative reals such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *with rate of divergence* r*. Then whenever* $\|x_n - q\|$ *is bounded above by some* $c > 0$*, we have* $\|x_n - q\| \to 0$*, and moreover, for any* $\varepsilon > 0$ *we have*

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ *is defined by*

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right), 2\int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)}\right)$$

and

$$\max\left\{f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right)\right\}$$

where f_1, f_2 *are rates of convergence for* $k_n, l_n \to 0$.

Theorem (= Theorem A$^+$ + Lemma B$^+$ + rate conversion)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, we have $\|x_n - q\| \to 0$, with rate of convergence

$$\|x_n - q\| \leq F^{-1}\left(2\Psi(c) - \sum_{i=0}^{n-2} \alpha_i\right)$$

where $F : (0,\infty) \to \mathbb{R}$ is any strictly increasing and continuous function satisfying

$$F(\varepsilon) \geq 2\Psi\left(\frac{\varepsilon}{2}\right) - \alpha \cdot \sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right)$$

and Ψ is given by

$$\Psi(s) := \int^{s} \frac{dt}{\psi(t)}$$

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4. Summary of further results

Further results I: d-weakly contractive mappings

Let X be a **uniformly smooth** Banach space, X^* be the dual of X, and $J : X \to X^*$ the normalized duality mapping i.e.

$$\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2$$

We call $\{A_n\}$ quasi asymptotically d-weakly contractive w.r.t. ψ and q with modulus σ if for any $\delta, c > 0$ we have

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\langle A_n x - q, J(A_n x - q)\rangle \leq \|x - q\|^2 - \psi(\|x - q\|) + \delta)$$

The sequence

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

converges to q, where we can construct a rate of convergence in the modulus of uniform smoothness for the space X.

This generalises and provides a rate of convergence for a theorem of [Chidume et al., 2002].

Further results II: Perturbed schemes

Suppose that $\{A_n\}$ with $A_n : E_n \to E$ are asymptotically weakly contractive w.r.t. ψ and q, and $\{x_n\}$ satisfies the perturbed scheme

$$x_{n+1} = Q_n((1-\alpha_n)x_n + \alpha_n A_n x_n)$$

where $Q_n : X \to E_{n+1}$ is a Sunny nonexpansive retraction. Then x_n converges to q, provided that X is uniformly smooth and

$$E_n \to E$$

w.r.t Hausdorff metric. Uses a formalisation of the Hausdorff distance first used in [Kohlenbach and Powell, 2020].

This generalises and provides a rate of convergence for a theorem of [Alber et al., 2003].

Summary

$\boxed{\text{space } X} + \boxed{\text{mapping } \{A_n\}} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence}}$

space	contraction mapping	algorithm
normed	ψ-weakly	Picard
normed	totally asymptotically ψ-weakly	Mann
normed	quasi asymptotically ψ-weakly	Mann
unif. smooth	quasi asymptotically d-weakly	Mann
unif. smooth	asymptotically ψ-weakly	perturbed Mann

In each case, we use the same reduction to the recursive inequality

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n \delta$$

for sufficiently large n, and provide explicit rates of convergence.

Rates of convergence for asymptotically weakly contractive mappings in normed spaces T. Powell and F. Wiesnet, **Numerical Functional Analysis and Optimization**, 2021.

Ideas for future work

A general study of recursive inequalities

Abstract recursive inequalities play a central role in nonlinear analysis, and a quantitative analysis of such inequalities has been crucial in many applied proof theory papers.

For instance, in [Kohlenbach and Powell, 2020] the following recursive inequality is studied:

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_{n+1}) + \alpha_n \gamma_n$$

for $\gamma_n \to 0$.

It would be interesting to have a general quantitative study of recursive inequalities:

- Bringing together known results and establishing new ones,
- Providing a repository of quantitative lemmas which could then be applied in concrete situations,
- Understanding precisely when we can expect rates of convergence,
- Make some inroads into formalizing applied proof theory!

The bigger picture

New areas for undercover proof theory:

- Analytic number theory.
- Computational algebra.
- Probability and machine learning.

Alber, Y., Chidume, C., and Zegeye, H. (2006).
Approximating fixed points of total asymptotically nonexpansive mappings.
Fixed Point Theory and Applications, 2006:1687–1812.

Alber, Y. and Guerre-Delabriere, S. (1997).
Principle of weakly contractive maps in Hilbert spaces.
In Gohberg, I. and Lyubich, Y., editors, *New Results in Operator Theory and its Applications*, volume 98, pages 7–22.

Alber, Y., Reich, S., and Yao, J.-C. (2003).
Iterative methods for solving fixed-point problems with nonself-mappings in Banach spaces.
Abstract and Applied Analysis, 2003(4).

Chidume, C., Zegeye, H., and Aneke, S. J. (2002).
Approximation of fixed points of weakly contractive nonself maps in Banach spaces.
Journal of Mathematical Analysis and Applications, 270:189–199.

Kohlenbach, U. and Powell, T. (2020).
Rates of convergence for iterative solutions of equations involving set-valued accretive operators.
Computers and Mathematics with Applications, 80:490–503.

Powell, T. and Wiesnet, F. (2021).
Rates of convergence for asymptotically weakly contractive mappings in normed spaces.
To appear: Numerical Functional Analysis and Optimization, 2021.