Some recent work in proof mining

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These slides will be available at
`https://t-powell.github.io/talks`

Applied proof theory (aka ‘proof mining’) in one slide:

Uses ideas and techniques from proof theory to analyse mathematical proofs and:

- Extract quantitative information (even when the proof is at first glance nonconstructive).
- Obtain generalisations of the original theorem through weakening/abstracting assumptions.
- Give deeper insights into theorems from ‘mainstream’ mathematics and provide a uniform framework through which different results can be brought together.

Aims of this talk:

- Present a recent application of proof theory in nonlinear analysis.
- Provide some general insight into how proof mining is done in practice.

Outline

1. A high level overview
2. A simple worked example
3. A first general result
4. Summary of further results and conclusion

We start with something familiar:

Throughout this talk, we work in a Banach space X.

A mapping $T : E \to E$ for $E \subseteq X$ is called *strongly contractive* (or often just a *contraction mapping*) if there exists $k \in [0, 1)$ such that $\forall x, y \in E$:

$$\|Tx - Ty\| \leq (1 - k)\,\|x - y\|$$

Theorem (Banach fixed point theorem)

If T is strongly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \frac{(1-k)^n}{k}\,\|x_1 - x_0\|$$

$$\boxed{\text{space } X} + \boxed{\text{mapping } T} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence to fixpoint}}$$

A generalisation of the Banach fixed point theorem:

A mapping $T : E \to E$ for $E \subseteq X$ is called ψ-*weakly contractive* if $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

In the case that $\psi(t) := kt$ then T is strongly contractive.

Theorem ([Alber and Guerre-Delabriere, 1997])

If T is weakly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

$$\boxed{\text{space } X} + \boxed{\text{mapping } T} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence to fixpoint}}$$

Example of a weakly contractive mapping

Define $X = \mathbb{R}$ and $T : [0,1] \to [0,1]$ by $Tx := \sin x$. Then we can show that

$$|\sin x - \sin y| \leq |x-y| - \frac{1}{8}|x-y|^3$$

and so sin is ψ-weakly contractive for $\psi(t) = \frac{1}{8}t^3$.

The unique fixpoint of sin is $x = 0$, and defining $x_{n+1} := \sin x_n$ we have $x_n \to 0$ with rate of convergence

$$x_n \leq \frac{1}{\sqrt{x_0^{-2} + \frac{n-1}{4}}}$$

(cf. [Alber and Guerre-Delabriere, 1997] for details).

A further generalisation:

A mapping $T : E \to E$ for $E \subseteq X$ is called totally asymptotically ψ-weakly contractive if $\psi, \phi : [0, \infty) \to [0, \infty)$ are nondecreasing functions with $\psi(0) = \phi(0) = 0$ and $\psi(t), \phi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

for $k_n, l_n \to 0$. In the case that $k_n = l_n := 0$ then T is ψ-weakly contractive.

Theorem (Adapted from [Alber et al., 2006])

Suppose that $E \subseteq X$ is convex, T is asymptotically ψ-weakly contractive and q is a fixpoint of T. Moreover, from any starting point x_0 define the sequence $\{x_n\}$ by

$$x_{n+1} = (1 - \alpha_n) x_n + \alpha_n T^n x_n$$

where $\{\alpha_n\}$ is some sequence of nonnegative reals with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\|x_n - q\|$ is bounded. Then $x_n \to q$.

A clear closed form expression for a rate of convergence is not given in [Alber et al., 2006].

space X + mapping T + algorithm $\{x_n\}$ $\Longrightarrow$ convergence to fixpoint

First objective: Define a general class of mappings of 'weakly contractive type'

Definition ([P. and Wiesnet, 2021])

A sequence of mappings $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Example. If T is totally asymptotically ψ-weakly contractive in the sense that

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

then $\{T^n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. any fixpoint of T with modulus

$$\sigma(\delta, c) := \max \left\{ f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right) \right\}$$

where f_1, f_2 are rates of convergence for $k_n, l_n \to 0$.

Second objective: Produce general convergence theorems

Theorem (Adapted from [P. and Wiesnet, 2021])

Suppose that $E \subseteq X$ is convex, $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ. Moreover, from any starting point x_0 define the sequence $\{x_n\}$ by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n A_n x_n$$

where $\{\alpha_n\} \in [0, \alpha]$ is some sequence of nonnegative reals with $\sum_{n=0}^{\infty} \alpha_n = \infty$. Suppose that $\|x_n - q\|$ is bounded by $c > 0$. Then $x_n \to q$, with rate of convergence

$$\|x_n - q\| \leq F^{-1}\left(2\Psi(c) - \sum_{i=0}^{n-2} \alpha_i\right)$$

where $F : (0, \infty) \to \mathbb{R}$ is any strictly increasing and continuous function satisfying

$$F(\varepsilon) \geq 2\Psi\left(\frac{\varepsilon}{2}\right) - \alpha \cdot \sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right)$$

and Ψ is given by

$$\Psi(s) := \int^{s} \frac{dt}{\psi(t)}$$

How are these results obtained?

- An analysis of the logical structure of key properties and assumptions.
- An analysis of the convergence proofs (which often use liminfs, convergent subsequences etc).
- A study of the relevant literature, identifying common patterns.

Outline

$T : E \to E$ is ψ-weakly contractive if $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(0) = 0$ and $\psi(t) > 0$ for $t > 0$, and $\forall x, y \in E$:

$$\|Tx - Ty\| \leq \|x - y\| - \psi(\|x - y\|)$$

Theorem (A)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then

$$\|x_{n+1} - q\| \leq \|x_n - q\| - \psi(\|x_n - q\|)$$

for all $n \in \mathbb{N}$.

Proof. We observe that

$$\begin{aligned} \|x_{n+1} - q\| &= \|Tx_n - q\| && \text{definition of } x_{n+1} \\ &= \|Tx_n - Tq\| && q \text{ a fixpoint of } T \\ &\leq \|x_n - q\| - \psi(\|x_n - q\|) && T \text{ is } \psi\text{-weakly contractive} \end{aligned}$$

Lemma (B)

Let $\{\mu_n\}$ be a sequence of nonnegative reals satisfying

$$\mu_{n+1} \leq \mu_n - \psi(\mu_n)$$

where $\psi : [0, \infty) \to [0, \infty)$ is a nondecreasing function with $\psi(t) > 0$ for $t > 0$. Then $\mu_n \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := \left\lceil \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \right\rceil$$

Proof. Suppose for contradiction that there exists $\varepsilon > 0$ such that $\mu_n > \varepsilon$ for all $n \in \mathbb{N}$. Observe that

$$\begin{aligned} 1 &\leq \frac{\mu_n - \mu_{n+1}}{\psi(\mu_n)} && (\text{definition of } \mu_n \text{ and } \psi(\mu_n) > 0) \\ &\leq \int_{\mu_{n+1}}^{\mu_n} \frac{dt}{\psi(t)} && (1/\psi(t) \text{ nonincreasing}) \end{aligned}$$

Proof (cont).
For any $N \in \mathbb{N}$ we have

$$\begin{aligned} N &= \sum_{i=0}^{N-1} 1 \\ &\leq \sum_{i=0}^{N-1} \int_{\mu_{n+1}}^{\mu_n} \frac{dt}{\psi(t)} \quad \text{(previous slide)} \\ &\leq \int_{\mu_N}^{\mu_0} \frac{dt}{\psi(t)} \quad (\mu_{n+1} < \mu_n) \\ &\leq \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \quad (\varepsilon < \mu_N) \end{aligned}$$

But this is false for

$$N := \left\lceil \int_{\varepsilon}^{\mu_0} \frac{dt}{\psi(t)} \right\rceil$$

and therefore there exists some $n \leq N$ such that $\mu_n \leq \varepsilon$. But then in particular, since

$$\mu_{n+1} \leq \mu_n - \psi(\mu_n) \leq \mu_n$$

it follows that $\mu_n \leq \varepsilon$ for all $n \geq N$.

Theorem (= Theorem A + Lemma B)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then $\|x_n - q\| \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := \left\lceil \int_{\varepsilon}^{\|x_0 - q\|} \frac{dt}{\psi(t)} \right\rceil$$

This is a perfectly satisfactory quantiative convergence theorem, where we provide a 'proof theorist's' rate of convergence for $\mu_n \to 0$ i.e. a function Φ such that

$$\forall \varepsilon > 0, \forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

Analysts, on the other hand, typically formulate a rate of convergence as a function f such that

$$\forall n(\mu_n \leq f(n))$$

where $f(n) \to 0$ as $n \to \infty$.

Rate conversion. We have shown that for any $\varepsilon > 0$ we have $\|x_n - q\| \le \varepsilon$ for

$$n \ge \left\lceil \int_{\varepsilon}^{\|x_0 - q\|} \frac{dt}{\psi(t)} \right\rceil$$

We now want to find for each $n \in \mathbb{N}$ some ε_n such that

$$\|x_n - q\| \le \varepsilon_n$$

This would work for any ε_n with

$$n - 1 < \int_{\varepsilon_n}^{\|x_0 - q\|} \frac{dt}{\psi(t)} = \Psi(\|x_0 - q\|) - \Psi(\varepsilon_n) \le n$$

so define ε_n such that

$$\Psi(\|x_0 - q\|) - \Psi(\varepsilon_n) = n$$

i.e.

$$\varepsilon_n := \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

Theorem (= Theorem A + Lemma B + rate conversion)

Suppose that T is ψ-weakly contractive and q is a fixpoint of T. Define $x_{n+1} := Tx_n$ for any starting point x_0. Then $\|x_n - q\| \to 0$, and moreover, for any $n \in \mathbb{N}$ we have

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

Now compare this to:

Theorem ([Alber and Guerre-Delabriere, 1997])

If T is weakly contractive then it possesses a fixpoint q. Moreover, from any starting point x_0 the sequence $\{x_n\}$ defined by $x_{n+1} := Tx_n$ converges to q, with rate of convergence

$$\|x_n - q\| \leq \Psi^{-1}(\Psi(\|x_0 - q\|) - n)$$

where Ψ is given by

$$\Psi(s) := \int^s \frac{dt}{\psi(t)}$$

General route to a convergence theorem

1. Reduce everything to a recursive inequality in terms of $\mu_n := \|x_n - q\|$.
2. Apply a general quantitative convergence theorem for this inequality.
3. Convert proof-theoretic rate into analyst's rate (optional, but essential if we want to compare with known bounds in simple cases).

Steps 2 and 3 can be done in a very general setting, so that in concrete cases, we only need to adapt Step 1!

Outline

A sequence $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t. q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Theorem (A^+)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, for any $\delta > 0$ and $n \geq \sigma(\delta, c)$ we have:

$$\|x_{n+1} - q\| \leq \|x_n - q\| - \alpha_n \psi(\|x_n - q\|) + \alpha_n \delta$$

Proof. We observe that for $n \geq \sigma(\delta, c)$

$$\begin{aligned}
\|x_{n+1} - q\| &= \|(1 - \alpha_n)(x_n - q) + \alpha_n(A_n x_n - q)\| && \text{(rearranging)} \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n \|A_n x_n - q\| && \text{(triangle inequality)} \\
&\leq (1 - \alpha_n)\|x_n - q\| + \alpha_n(\|x_n - q\| - \psi(\|x_n - q\|) + \delta) && \text{(property of } \{A_n\}) \\
&= \|x_n - q\| - \alpha_n \psi(\|x_n - q\|) + \alpha_n \delta
\end{aligned}$$

Lemma (B^+)

Let $\{\mu_n\}$ *be a sequence of nonnegative reals such that for any* $\delta > 0$ *we have*

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n \delta$$

for all $n \geq \sigma(\delta)$*, where:*

- $\psi : [0, \infty) \to [0, \infty)$ *is a nondecreasing function with* $\psi(t) > 0$ *for* $t > 0$*;*
- $\{\alpha_n\} \subset [0, \alpha]$ *is a sequence of nonnegative real numbers such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *with rate of divergence* $r : (0, \infty) \times (0, \infty) \to \mathbb{N}$ *i.e.*

$$\forall N \in \mathbb{N}, x > 0 \left(\sum_{n=N}^{r(N,x)} \alpha_n > x \right)$$

Then $\mu_n \to 0$*, and moreover, for any* $\varepsilon > 0$ *we have*

$$\forall n \geq \Phi(\varepsilon)(\mu_n \leq \varepsilon)$$

where Φ *is defined by*

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2} \min\left\{ \psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha} \right\} \right), 2\int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)} \right)$$

and c *is an upper bound for* $\{\mu_n\}$*.*

Theorem (= Theorem A^+ + Lemma B^+)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals such that $\sum_{n=0}^{\infty} \alpha_n = \infty$ with rate of divergence r. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, we have $\|x_n - q\| \to 0$, and moreover, for any $\varepsilon > 0$ we have

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ is defined by

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right), 2\int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)}\right)$$

Recall from earlier...

Definition ([P. and Wiesnet, 2021])

A sequence of mappings $\{A_n\}$ with $A_n : E \to E$ is quasi asymptotically ψ-weakly contractive w.r.t q and with modulus σ if for all $\delta, c > 0$ and $x, y \in E$:

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\|A_n x - q\| \leq \|x - q\| - \psi(\|x - q\|) + \delta)$$

Example. If T is totally asymptotically ψ-weakly contractive in the sense that

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

then $\{T^n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. any fixpoint of T with modulus

$$\sigma(\delta, c) := \max\left\{f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right)\right\}$$

where f_1, f_2 are rates of convergence for $k_n, l_n \to 0$.

Corollary (Quantitative version of [Alber et al., 2006])

Suppose that $T : E \to E$ *is quasi totally asymptotically* ψ*-weakly contractive in the sense that*

$$\|T^n x - T^n y\| \leq \|x - y\| - \psi(\|x - y\|) + k_n \phi(\|x - y\|) + l_n$$

for $k_n, l_n \to 0$*, that* q *is a fixpoint of* T *and that the sequence* $\{x_n\}$ *satisfies*

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n T^n x_n$$

for $\{\alpha_n\}$ *a sequence of nonnegative reals such that* $\sum_{n=0}^{\infty} \alpha_n = \infty$ *with rate of divergence* r*. Then whenever* $\|x_n - q\|$ *is bounded above by some* $c > 0$*, we have* $\|x_n - q\| \to 0$*, and moreover, for any* $\varepsilon > 0$ *we have*

$$\forall n \geq \Phi(\varepsilon)(\|x_n - q\| \leq \varepsilon)$$

where Φ *is defined by*

$$\Phi(\varepsilon) := r\left(\sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right), 2\int_{\varepsilon/2}^{c} \frac{dt}{\psi(t)}\right)$$

and

$$\max\left\{f_1\left(\frac{\delta}{2\phi(c)}\right), f_2\left(\frac{\delta}{2}\right)\right\}$$

where f_1, f_2 *are rates of convergence for* $k_n, l_n \to 0$.

Theorem (= Theorem A$^+$ + Lemma B$^+$ + rate conversion)

Suppose that $\{A_n\}$ is quasi asymptotically ψ-weakly contractive w.r.t. q and σ, and that the sequence $\{x_n\}$ satisfies

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

for $\{\alpha_n\}$ a sequence of nonnegative reals such that $\sum_{n=0}^{\infty} \alpha_n = \infty$. Then whenever $\|x_n - q\|$ is bounded above by some $c > 0$, we have $\|x_n - q\| \to 0$, with rate of convergence

$$\|x_n - q\| \leq F^{-1}\left(2\Psi(c) - \sum_{i=0}^{n-2} \alpha_i\right)$$

where $F : (0, \infty) \to \mathbb{R}$ is any strictly increasing and continuous function satisfying

$$F(\varepsilon) \geq 2\Psi\left(\frac{\varepsilon}{2}\right) - \alpha \cdot \sigma\left(\frac{1}{2}\min\left\{\psi\left(\frac{\varepsilon}{2}\right), \frac{\varepsilon}{\alpha}\right\}, c\right)$$

and Ψ is given by

$$\Psi(s) := \int^{s} \frac{dt}{\psi(t)}$$

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Further results I: d-weakly contractive mappings

Let X be a **uniformly smooth** Banach space, X^* be the dual of X, and $J : X \to X^*$ the normalized duality mapping i.e.

$$\langle x, Jx \rangle = \|x\|^2 = \|Jx\|^2$$

We call $\{A_n\}$ quasi asymptotically d-weakly contractive w.r.t. ψ and q with modulus σ if for any $\delta, c > 0$ we have

$$\|x - q\| \leq c \implies \forall n \geq \sigma(\delta, c)(\langle A_n x - q, J(A_n x - q) \rangle \leq \|x - q\|^2 - \psi(\|x - q\|) + \delta)$$

The sequence

$$x_{n+1} = (1 - \alpha_n)x_n + \alpha_n A_n x_n$$

converges to q, where we can construct a rate of convergence in the modulus of uniform smoothness for the space X.

This generalises and provides a rate of convergence for a theorem of [Chidume et al., 2002].

Further results II: Perturbed schemes

Suppose that $\{A_n\}$ with $A_n : E_n \to E$ are asymptotically weakly contractive w.r.t. ψ and q, and $\{x_n\}$ satisfies the perturbed scheme

$$x_{n+1} = Q_n((1-\alpha_n)x_n + \alpha_n A_n x_n)$$

where $Q_n : X \to E_{n+1}$ is a Sunny nonexpansive retraction. Then x_n converges to q, provided that X is uniformly smooth and

$$E_n \to E$$

w.r.t Hausdorff metric. Uses a formalisation of the Hausdorff distance first used in [Kohlenbach and Powell, 2020].

This generalises and provides a rate of convergence for a theorem of [Alber et al., 2003].

Summary

$$\boxed{\text{space } X} + \boxed{\text{mapping } \{A_n\}} + \boxed{\text{algorithm } \{x_n\}} \implies \boxed{\text{convergence}}$$

space	contraction mapping	algorithm
normed	ψ-weakly	Picard
normed	totally asymptotically ψ-weakly	Mann
normed	quasi asymptotically ψ-weakly	Mann
unif. smooth	quasi asymptotically d-weakly	Mann
unif. smooth	asymptotically ψ-weakly	perturbed Mann

In each case, we use the same reduction to the recursive inequality

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_n) + \alpha_n \delta$$

for sufficiently large n, and provide explicit rates of convergence.

My priority for future work

Abstract recursive inequalities play a central role in nonlinear analysis, and a quantitative analysis of such inequalities has been crucial in many applied proof theory papers.

For instance, in [Kohlenbach and Powell, 2020] the following recursive inequality is studied:

$$\mu_{n+1} \leq \mu_n - \alpha_n \psi(\mu_{n+1}) + \alpha_n \gamma_n$$

for $\gamma_n \to 0$.

It would be interesting to have a general quantitative study of recursive inequalities:

- Bringing together known results and establishing new ones,
- Providing a repository of quantitative lemmas which could then be applied in concrete situations.

THANK YOU!

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