On the computational content of Zorn's lemma

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LOGIC IN COMPUTER SCIENCE (LICS '20)

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These slides are available at
`https://t-powell.github.io/talks`

Before I start...

This talk is designed to **complement** the paper. Specifically I will focus on

- the mathematical and historical context
- outlining the main problem
- providing a **brief sketch** of the central ideas.

Any technical content is presented in a **very informal** manner.

If you have any questions, please send me an email!

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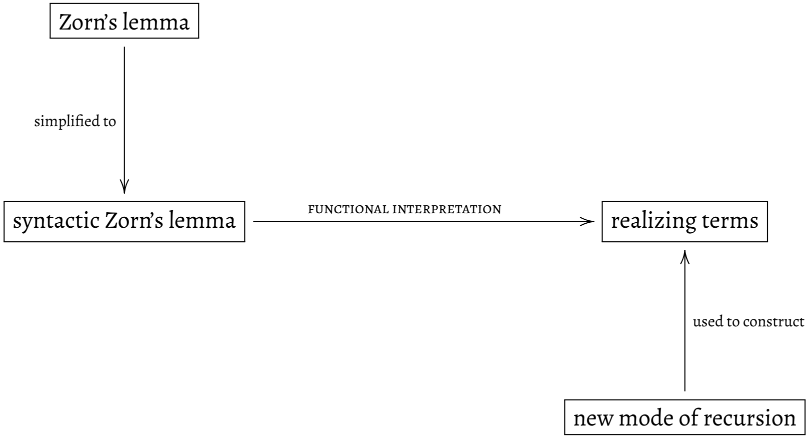

Zorn's lemma

Theorem (Zorn's lemma)

Let $(S, <)$ be a nonempty set equipped with a (strict) partial order.

Suppose that every nonempty chain (= totally ordered subset) $\gamma \subseteq S$ has an upper bound in S, i.e. there exists some $u \in S$ such that $x \leq u$ for all $x \in \gamma$.

Then S contains a maximal element i.e. there exists some $m \in S$ such that $\neg(m < x)$ for any $x \in S$.

In 'ordinary' mathematics, Zorn's lemma is typically used to 'build' objects in stages e.g.

- every set has a well ordering,
- every vector space has a basis,
- every nontrivial ring has a maximal ideal.

Over ZF we have

$$\text{Zorn's lemma} \Leftrightarrow \text{Axiom of choice}$$

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Translates each formula A in HA^ω (Heyting arithmetic in all finite types) to one of the form $\exists x \forall y\, A_D(x,y)$, where x, y are (possibly empty) tuples of variables and

- $A \Leftrightarrow \exists x \forall y\, A_D(x,y)$
- $A_D(x,y)$ is quantifier-free (and hence decidable).

Theorem. (essentially Goedel 1958)

1. Suppose that $\mathrm{HA}^\omega \vdash A$. Then from the proof of A we can extract a term t of System T such that $T \vdash \forall y\, A_D(t,y)$.
2. Suppose that $\mathrm{PA}^\omega \vdash A$. Then we can extract a term t of System T such that $T \vdash \forall y\; (A^N)_D(t,y)$ where A^N is the negative translation of A.

Applications of the functional interpretation include:

(a) Relative consistency proofs i.e. $\mathrm{Con}(T) \Rightarrow \mathrm{Con}(\mathrm{PA}^\omega)$,

(b) Program extraction: if $\mathrm{PA}^\omega \vdash \forall x \exists y\; P(x,y)$ we can extract a primitive recursive t such that $T \vdash \forall x\; P(x, tx)$. See also the **proof mining** program.

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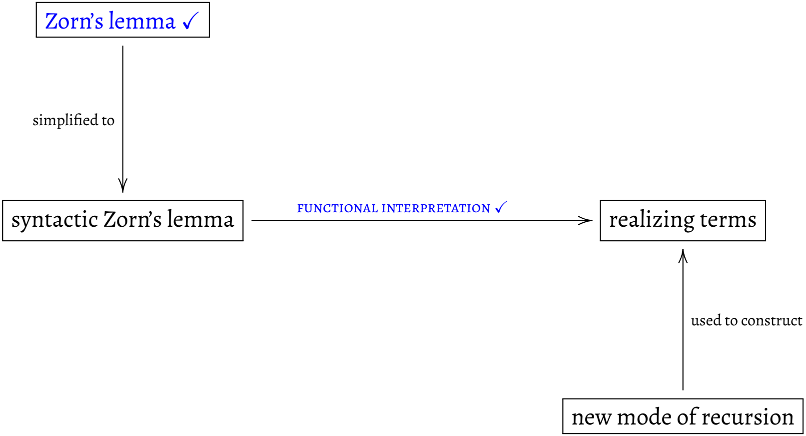
The paper in one slide



Related work

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- **Berardi, Bezem and Coquand 1998:** Symmetric mode of recursion for interpreting axiom of countable choice (BBC functional), based making recursive calls to one-element extensions of countable sets.
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A syntactic formulation of Zorn's lemma

Zorn's lemma: nonemptyness + chain bounded ⇒ maximal element

Syntactic Zorn's lemma (in language of PA^ω)

$$\mathrm{SZL}_< \quad : \quad \underbrace{\exists x\, P(x)}_{\text{nonemptyness}} \Rightarrow \underbrace{\exists y (P(y) \wedge \forall z > y \neg P(z))}_{\text{maximal element}}$$

where:

- $<$ need not be wellfounded, but has specific logical structure,
- $P(x)$ is 'piecewise' (only looks at a 'finite parts' of x).

$PA^\omega + \mathrm{SZL}_<$ is valid in models where (the interpretation of) $<$ is chain bounded.

SZL is a special instance of Zorn's lemma, and its validity depends on the choice of parameters. However, there is a simple instatiation $<_0$ of $<$ for which

$$PA^\omega \vdash \mathrm{SZL}_{<_0} \Leftrightarrow \text{dependent choice}$$

and so a special case of SZL has the strength of full classical analysis (in the sense of Spector).

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Simple example: Subsets of $\mathbb{N}$

Suppose that $x, y, z : \mathsf{nat} \to \mathsf{bool}$ represent characteristic functions of subsets of natural numbers, and define

$$x < y :\Leftrightarrow x \subset y.$$

Then $\mathrm{SZL}_{\subset}$ becomes

$$\exists x\, P(x) \Rightarrow \exists y (P(y) \wedge \forall z \supset y\, \neg P(z))$$

where here $P(x)$ is piecewise if it can be written as

$$P(x) :\equiv \forall n\, Q([x_0, \ldots, x_{n-1}]).$$

$\mathrm{SZL}_{\subset}$ is closely related to Berger's **update induction**. Can be used to give direct proofs to a number of important results e.g.

$$\mathrm{PA}^{\omega} + \mathrm{SZL}_{\subset} \vdash \text{all nontrivial countable commutative rings have a maximal ideal}$$

Can show that both $\mathcal{S}^{\omega}$ and $\mathcal{C}^{\omega}$ are models of $\mathrm{PA}^{\omega} + \mathrm{SZL}_{\subset}$.

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$$P(x) :\equiv \forall n\, Q([x_0, \ldots, x_{n-1}]).$$

$\mathrm{SZL}_{\subset}$ is closely related to Berger's **update induction**. Can be used to give direct proofs to a number of important results e.g.

$\mathrm{PA}^{\omega} + \mathrm{SZL}_{\subset} \vdash$ all nontrivial countable commutative rings have a maximal ideal

Can show that both $\mathcal{S}^{\omega}$ and $\mathcal{C}^{\omega}$ are models of $\mathrm{PA}^{\omega} + \mathrm{SZL}_{\subset}$.

A computational counterpart to $SZL_{\subset}$

What form of recursion can we associate with

$$\exists x\, P(x) \Rightarrow \exists y (P(y) \wedge \forall z \supset y\, \neg P(z))?$$

The most obvious is a recursor over $\supset$ i.e.

$$\Phi f(x) =_\rho fx(\lambda y \,.\, \Phi f(y) \text{ if } y \supset x).$$

This tricky because $x \supset y$ is not decidable, but can reformulate as

$$\Phi f(x) =_\rho fx(\lambda n^{\mathsf{nat}}, y \,.\, \Phi f(x \cup y) \text{ if } n \in y \text{ and } n \notin x).$$

But is $\Phi f(x)$ well-defined?

Theorem (sketch)

Suppose that $\rho := \mathsf{nat}$. Then Φf is a total object in the model of partial continuous functionals since the value of $\Phi f(x)$ only depends on a finite part of the input x.

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A problem with higher types

Let us now consider

$$\Phi f(x) =_\rho fx(\lambda n^{\mathsf{nat}}, y \,.\, \Phi f(x \cup y) \text{ if } n \in y \text{ and } n \notin x).$$

in the case $\rho := \mathsf{nat} \to \mathsf{nat}$. Define f by

$$fxp := \lambda n \,.\, 1 + p(n, \{n\})(n+1).$$

Then for $k := \Phi f(\emptyset)0$ we have

$$k = 1 + \Phi f(\{0\})(1) = 2 + \Phi f(\{0,1\})(2)$$
$$\ldots = k + 1 + \Phi f(\{0, \ldots, k\})(k+1) > k$$

so the defining equation of Φ is inconsistent with PA^ω.

Problem

We need a form of recursion which is valid for all output types ρ to solve the functional interpretation of $\mathrm{SZL}_\subset$.

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We need a form of recursion which is valid for all output types ρ to solve the functional interpretation of $\mathrm{SZL}_\subset$.

Solution

We define an alternative form of recursion based on the idea of a 'truncation'. Let

$$\Psi\omega f(x) =_\rho f\{x\}_\omega(\lambda n, y \,.\, \Psi f(\{x\}_\omega \cup y) \text{ if } n \in y \text{ and } n \notin \{x\}_\omega)$$

where, very roughly, $\{x\}_\omega$ is an operation which truncates the input x and forces only a finite part to be relevant.

Idea: Piecewiseness of totality predicate is controlled syntactically.

Theorem (sketch)

Under reasonable assumptions, Ψf defines a total continuous functional for any output type ρ.

Main contributions (sketch)

- A generalisation of Ψ above to $\Psi_<$, together with precise conditions on $<$ which guarantee that $\Psi_<$ defines a total continuous functional.
- Proof that there is a term definable in $\Psi_<$ which realizes the functional interpretation of $\mathrm{SZL}^N_<$, hence

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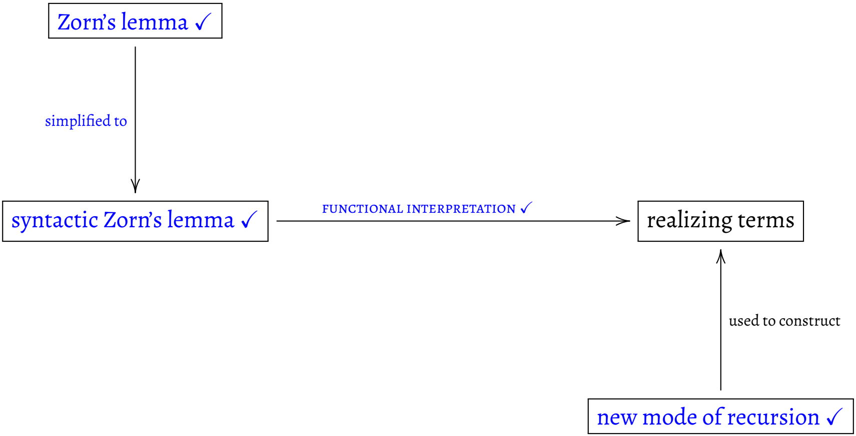
The paper in one slide



THANK YOU!