Quantitative Tauberian theorems

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MATHEMATICAL LOGIC: PROOF THEORY, CONSTRUCTIVE MATHEMATICS

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These slides are available at
`https://t-powell.github.io/talks`

Introduction

Abel's theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$\sum_{i=0}^{\infty} a_i = s$$

it follows that

$$F(x) \to s \quad \text{as} \quad x \nearrow 1.$$

This is a classical result in elementary analysis called **Abel's theorem** (N.b. it also holds in the complex setting). You can use it to e.g. prove that

$$\sum_{i=0}^{\infty} \frac{(-1)^i}{i+1} = \log(2).$$

Does the converse of Abel's theorem hold?

NO.

For a counterexample, define $F : (-1, 1) \to \mathbb{R}$ by

$$F(x) = \frac{1}{1+x} = \sum_{i=0}^{\infty} (-1)^i x^i$$

Then

$$F(x) \to \frac{1}{2} \quad \text{as} \quad x \nearrow 1$$

but

$$\sum_{i=0}^{\infty} (-1)^i \quad \text{does not converge}$$

Tauber's theorem fixes this

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$F(x) \to s \quad \text{as} \quad x \nearrow 1 \quad \text{AND} \quad |na_n| \to 0$$

it follows that

$$\sum_{i=0}^{\infty} a_i = s$$

This is **Tauber's theorem**, proven in 1897 by Austrian mathematician Alfred Tauber (1866 - 1942).

Tauberian theorems

The basic structure of Tauber's theorem is:

$$\text{Let} \quad F(x) = \sum_{i=0}^{\infty} a_i x^i$$

Then if we know

(A) Something about the behaviour of $F(x)$ as $x \nearrow 1$

(B) Something about the growth of $\{a_n\}$ as $n \to \infty$

Then we can conclude

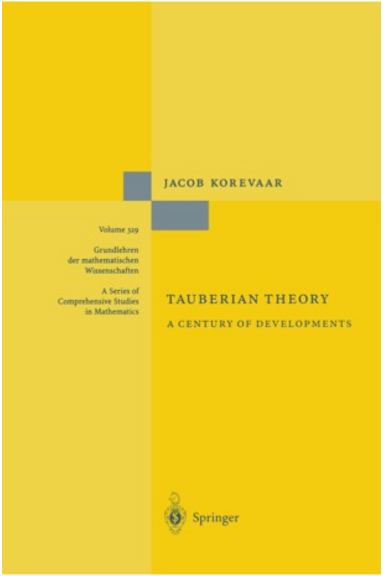
(C) Something about the convergence of $\sum_{i=0}^{\infty} a_i$.

This basic idea has been **considerably generalised** e.g. for

$$F(s) := \int_1^{\infty} a(t) t^{-s} \, \mathrm{d}t$$

and has grown into an area of research known as **Tauberian Theory**.

There is now a whole textbook (published 2004, 501 pages)



Tauberian theorems have an interesting structure

$$\boxed{\text{convergence}} + \boxed{\text{growth condition}} \Rightarrow \boxed{\text{convergence}}$$

Can we give a quantitative interpretation of these theorems e.g.

$$\boxed{\text{rate of convergence}} + \boxed{\text{quantitative growth condition}} \Rightarrow \boxed{\text{rate of convergence}}$$

In many cases, this would seem to pose a real challenge, as the proofs of Tauberian theorems are typically based on complicated analytic techniques.

Remainder of the talk:

1. Some very simple results (published)
2. Some slightly less simple results (unpublished)
3. Some rough ideas for the future (speculation)

Simple results

Cauchy variants of Abelian and Tauberian theorems

From now on, $\{a_n\}$ is a sequence of reals, $F(x) := \sum_{i=0}^{\infty} a_i x^i$ and $s_n := \sum_{i=0}^{n} a_i$.

The following may well feature in some form elsewhere in the literature...

Theorem (Abel's theorem, Cauchy variant)

Suppose that

- $\{s_n\}$ *is Cauchy,*
- $\{x_m\} \in [0, 1)$ *satisfies* $\lim_{m\to\infty} x_m = 1$.

Then $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$.

Theorem (Tauber's theorem, Cauchy variant)

Suppose that

- $\{F(v_m)\}$ *is Cauchy, where* $v_m := 1 - \frac{1}{m}$,
- $a_n = o(1/n)$.

Then $\lim_{m,n\to\infty} |F(v_m) - s_n| = 0$.

Theorem (Finite Abel's theorem, P. 2020)

Let $\{a_n\}$ and $\{x_k\}$ be arbitrary sequences of reals, and $L \in \mathbb{N}$ a bound for $\{|s_n|\}$. Fix some $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. Suppose that $N_1, N_2 \in \mathbb{N}$ and $p \geq 1$ are such that

$$|s_i - s_n| \leq \frac{\varepsilon}{4} \quad \text{and} \quad \frac{1}{p} \leq 1 - x_m \leq \frac{\varepsilon}{8LN_1}$$

for all $i, n \in [N_1; \max\{N + g(N), l\}]$ and all $m \in [N_2; N + g(N)]$ where

$$N := \max\{N_1, N_2\} \quad \text{and} \quad l := p \cdot \left\lceil \log\left(\frac{8Lp}{\varepsilon}\right)\right\rceil$$

Then we have $|F(x_m) - s_n| \leq \varepsilon$ for all $m, n \in [N; N + g(N)]$.

Simple application of Gödel's functional interpretation to textbook proof of Abel's theorem.

Nothing deep

Corollary: A quantitative Abel's theorem

Suppose that $|s_n| \leq L$ for all $n \in \mathbb{N}$ and moreover

(A) ϕ is a rate of Cauchy convergence for $\{s_n\}$, or equivalently

$$\forall \varepsilon > 0 \, \forall m, n \geq \phi(\varepsilon) \left(\left| \sum_{i=m}^{n} a_i \right| \leq \varepsilon \right)$$

(B) ψ is a rate of convergence for $x_m \nearrow 1$.

Then a rate of convergence for $\lim_{m,n\to\infty} |F(x_m) - s_n| = 0$ is given by

$$\Phi_{L,\phi,\psi}(\varepsilon) := \max \left\{ \phi(\varepsilon/4), \psi \left(\frac{\varepsilon}{8L\phi(\varepsilon/4)} \right) \right\}$$

A finitization of Tauber's theorem

Theorem (Finite Tauber's theorem, P. 2020)

Let $\{a_n\}$ be an arbitrary sequence of reals, and L a bound for $\{|a_n|\}$. Define $v_m := 1 - \frac{1}{m}$, and fix some $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$. Suppose that $N_1, N_2 \in \mathbb{N}$ are such that

$$i|a_i| \leq \frac{\varepsilon}{8} \ \ \textit{and} \ \ |F(v_m) - F(v_n)| \leq \frac{\varepsilon}{4}$$

for all $i \in [N_1; l]$ and all $m, n \in [N_2; N + g(N)]$ where

$$N := \max\left\{\frac{2LN_1^2}{\varepsilon}, N_2\right\} \ \ \textit{and} \ \ p \cdot \left\lceil \log\left(\frac{4Lp}{\varepsilon}\right)\right\rceil \ \ \textit{for} \ \ p := N + g(N)$$

Then we have $|F(v_m) - s_n| \leq \varepsilon$ for all $m, n \in [N; N + g(N)]$.

Corollary: A quantitative Tauber's theorem

Suppose that $|a_n| \leq L$ for all $n \in \mathbb{N}$ and moreover

(A) ϕ is a rate of Cauchy convergence for $n|a_n| \to 0$

(B) ψ is a rate of convergence for $\{F(1 - \frac{1}{m})\}$

Then a rate of convergence for $\lim_{m,n\to\infty} |F(1 - \frac{1}{m}) - s_n| = 0$ is given by

$$\Phi_{L,\phi,\psi}(\varepsilon) := \max\left\{\frac{2L\phi(\varepsilon/8)^2}{\varepsilon}, \psi(\varepsilon/4)\right\}$$

Again, nothing deep in any of the above results, but they can be used in conjunction with known results from proof mining to generate concrete quantitative lemmas:

Lemma

Let $\{a_n\}$ be a sequence of positive reals whose partial sums $\{s_n\}$ are bounded above. Then $\lim_{m,n\to\infty} |F(1-\frac{1}{m}) - s_n| = 0$.

Note: If $\{s_n\}$ is a Specker sequence then there is no computable rate of convergence for $\lim_{m,n\to\infty} |F(1-\frac{1}{m}) - s_n| = 0$.

Lemma

Let $\{a_n\}$ be a sequence of positive reals and L a bound for the partial sums $\{s_n\}$. Then for any $\varepsilon \in \mathbb{Q}_+$ and $g : \mathbb{N} \to \mathbb{N}$ we have

$$\exists N \leq \Gamma_L(\varepsilon, g)\ \forall m, n \in [N, N + g(N)]\ (|F(1-\tfrac{1}{m}) - s_n| \leq \varepsilon)$$

for $\Gamma_L(\varepsilon, g)$ given as follows:

- $\Gamma_L(\varepsilon, g) := \left\lceil \dfrac{8Lf^{(\lceil 4L/\varepsilon \rceil)}(0)}{\varepsilon} \right\rceil$
- $f(a) := p_a \cdot \left\lceil \log\left(\frac{8Lp_a}{\varepsilon}\right) \right\rceil$
- $p_a := \tilde{g}\left(\left\lceil \frac{8La}{\varepsilon} \right\rceil\right)$ *for* $\tilde{g}(x) := x + g(x)$.

These results (and more) appear in:

A note on the finitization of Abelian and Tauberian theorems

Thomas Powell

Abstract

We present finitary formulations of two well known results concerning infinite series, namely Abel's theorem, which establishes that if a series converges to some limit then its Abel sum converges to the same limit, and Tauber's theorem, which presents a simple condition under which the converse holds. Our approach is inspired by proof theory, and in particular Gödel's functional interpretation, which we use to establish quantitative version of both of these results.

1 Introduction

In an essay of 2007 [17] (later published as part of [18]) T. Tao discussed the so-called *correspondence principle* between 'soft' and 'hard' analysis, whereby many *infinitary* notions from analysis can be given an equivalent *finitary* formulation. An important instance of this phenomenon is provided by the simple concept of Cauchy convergence of a sequence $\{c_n\}$:

$$\forall \varepsilon > 0\ \exists N\ \forall m, n \geq N\ (|c_m - c_n| \leq \varepsilon).$$

This corresponds to the finitary notion of $\{c_n\}$ being *metastable*, which is given by the following formula:

$$\forall \varepsilon > 0\ \forall g : \mathbb{N} \to \mathbb{N}\ \exists N\ \forall m, n \in [N; N + g(N)]\ (|c_m - c_n| \leq \varepsilon), \tag{1}$$

where $[N; N+k] := \{N, N+1, \ldots, N+k-1, N+k\}$. Roughly speaking, a sequence $\{c_n\}$ is metastable if for any given error $\varepsilon > 0$ it contains a finite regions of stability of any 'size', where size is represented by the function $g : \mathbb{N} \to \mathbb{N}$.

The equivalence of Cauchy convergence and metastability is established via purely logical reasoning, and indeed, as was quickly observed, the correspondence principle as presented in [17] has deep connections with proof theory. More specifically, the finitary variant of an infinitary statement is typically closely related to its *classical Dialectica interpretation* [1], which provides a general method for obtaining quantitative versions of mathematical theorems.

Finitary formulations of infinitary properties play a central role in the *proof mining* program developed by U. Kohlenbach from the early 90s [7]. Here, it is often the case that a given mathematical theorem has, in general, no computable realizer (for Cauchy convergence this is demonstrated by the existence of so-called *Specker sequences* [16], which will be discussed further in Section 3). On the other hand, the corresponding finitary formulation can typically not only be realized, but a realizer can be directly extracted from a proof that the original property holds. The extraction of a computable bound $\Omega(\varepsilon, g)$ on N in (1) – a so-called *rate*

Less simple results

Tauber's theorem was first extended by Littlewood (1911)

THE CONVERSE OF ABEL'S THEOREM ON POWER SERIES

By J. E. Littlewood.

[Received September 28th, 1910.—Read November 10th, 1910.—
Revised December, 1910.*]

Introduction.

Abel's theorem states that if $\sum_0^\infty a_n$ is convergent, then $\lim \sum_0^\infty a_n x^n$ exists as $x \to 1$ by real values, and is equal to Σa_n. The converse theorem, however, that the existence of $\lim_{x \to 1} \Sigma a_n x^n$ implies the convergence of Σa_n, is very far from being true; for example, either the Cesàro or the Borel summability of Σa_n suffices for the existence of Abel's limit. It is known, however, that the existence of this limit, *combined with certain conditions satisfied by the a's*, does imply the convergence of Σa_n. Three such sets of conditions, for example, are:

(*a*)† the *a*'s are all positive;

(*b*) the order of a_n has a certain upper limit;

(*c*)‡ the function $\Sigma a_n x^n$ is regular at the point $x = 1$ and $a_n \to 0$.

In the present paper we are concerned with the problems arising out of case (*b*), where the only additional restriction on the *a*'s is an upper limit to the order of a_n. The theorem of this case is due to M. Tauber.§ The result is remarkable and apparently paradoxical in view of Abel's theorem, for it may be expressed roughly by saying that if Σa_n is not

Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that the power series

$$F(x) := \sum_{i=0}^{\infty} a_i x^i$$

converges on $|x| < 1$. Then whenever

$$F(x) \to s \quad \text{as} \quad x \nearrow 1 \quad \text{AND} \quad |na_n| \leq C$$

for some constant C, it follows that

$$\sum_{i=0}^{\infty} a_i = s$$

One of Littlewood's first major results. In *A Mathematical Education* he writes (of this period)

> *" On looking back this time seems to me to mark my arrival at a reasonably assured judgement and taste, the end of my "education". I soon began my 35-year collaboration with Hardy."*

One of first papers of this collaboration (1914):

TAUBERIAN THEOREMS CONCERNING POWER SERIES AND DIRICHLET'S SERIES WHOSE COEFFICIENTS ARE POSITIVE*

By G. H. HARDY and J. E. LITTLEWOOD.

[Received October 3rd, 1913.—Read November 13th, 1913.]

1. The general nature of the theorems contained in this paper resembles that of the "Tauberian" theorems which we have proved in a series of recent papers.† They have, however, a character of their own, in that they are concerned primarily with series of positive terms.

Let

$$f(x) = \Sigma a_n x^n$$

be a power series convergent for $|x| < 1$. We shall consider only positive values of x less than 1.

Let

$$s_n = a_0 + a_1 + \ldots + a_n,$$

$$L(u) = (\log u)^{a_1} (\log\log u)^{a_2} \ldots,$$

where the a's are real. Then it is known that, *if*

$$s_n \sim A n^a L(n),$$

where $A \neq 0$, *as* $n \to \infty$, *the indices* $a, a_1, a_2, \ldots$ *being such that* $n^a L(n)$ *tends to a positive limit or to infinity, then*

The Hardy-Littlewood Tauberian theorem

Let $\{a_n\}$ be a sequence of reals, and suppose that $\sum_{i=0}^{\infty} a_i x^i$ converges for $|x| < 1$. Then whenever

$$(1-x)\sum_{i=0}^{\infty} a_i x^i \to s \quad \text{as} \quad x \nearrow 1 \;\; \text{AND} \;\; a_n \geq -C$$

for some constant C, it follows that

$$\frac{1}{n}\sum_{i=0}^{n} a_i \to s \quad \text{as} \quad n \to \infty$$

They later used this result to give a new proof of the *prime number theorem*:

$$\pi(x) \sim \frac{x}{\log(x)}$$

A quantitative analysis of the Littlewood Tauberian theorem

Theorem (Finite Littlewood theorem, (unpublished))

Suppose that $\{a_n\}$ *satisfies* $n|a_n| \leq C$ *and* L *is a bound for* $|F(x)|$ *on* $(0,1)$. *Then there are constants* K_1 *and* K_2 *such that whenever* N_1 *satisfies*

$$\left| \sum_{k=0}^{\infty} a_k (e^{-ik/m} - e^{-jk/n}) \right| \leq \frac{\varepsilon}{4K_2^{C/\varepsilon}}$$

for all $(i,m),(j,n) \in [1;d] \times [dN_1; dN + g(dN)]$, *where*

$$N := \left\lceil \frac{4LK_2^{C/\varepsilon}}{\varepsilon} \right\rceil \cdot N_1 \quad \text{and} \quad d := \frac{K_1 C}{\varepsilon}$$

then we have $\left| \sum_{k=0}^{\infty} a_k e^{-k/m} - \sum_{k=0}^{n-1} a_k \right| \leq \varepsilon$ *for all* $m,n \in [N; N+g(N)]$.

Not quite as simple as Tauber's theorem: Requires in particular quantitative results bounding degree and coefficients of approximating polynomials.

Corollary: A quantitative Littlewood's Tauberian theorem (unpublished)

Suppose L is a bound for $|F(x)|$ on $(0, 1)$, and moreover

(A) $\phi : (0, \varepsilon) \to \mathbb{N}$ is a rate of Cauchy convergence for $F(x)$ in the sense that

$$\forall \varepsilon > 0\ \forall x, y \in [e^{-1/\phi(\varepsilon)}, 1)\ (|F(x) - F(y)| \leq \varepsilon).$$

(B) C is such that $\{a_n\}$ satisfies $n|a_n| \leq C$

Then a rate of Cauchy convergence $\psi : (0, \infty) \to \mathbb{N}$ for $\{s_n\}$ i.e.

$$\forall \varepsilon > 0\ \forall n \geq \psi(\varepsilon)\ (|s_m - s_n| \leq \varepsilon)$$

is given by

$$\psi_{C,L,\phi}(\varepsilon) := Lu \cdot \phi\left(\frac{1}{u}\right) \text{ for } u := \left\lceil \frac{D^{C/\varepsilon}}{\varepsilon} \right\rceil$$

for a suitable constant D.

Rates of convergence for Littlewood's Tauberian theorem have already been studied

BEST L_1 APPROXIMATION AND THE REMAINDER IN LITTLEWOOD'S THEOREM [1])

BY

JACOB KOREVAAR

(Communicated by Prof. H. D. KLOOSTERMAN at the meeting of March 28, 1953)

1. *Introduction and results.* Let $f(x)$ be continuous on $a \leqslant x \leqslant b$ and satisfy a LIPSCHITZ condition of order 1:

(1. 1) $\quad |f(x_1) - f(x_2)| \leqslant A|x_1 - x_2|$ for all x_1, x_2 on $a \leqslant x \leqslant b$.

D. JACKSON [2] has shown that for such an $f(x)$ there are a constant D and a sequence of polynomials $p_n(x)$ of degree n, $n = 1, 2, \ldots$, such that

$$\max_{a \leqslant x \leqslant b} |f(x) - p_n(x)| < D/n.$$

In this paper we consider approximation to functions $f(x)$ which are continuous on $a \leqslant x \leqslant b$ except for a finite number of jump discontinuities, and which satisfy a LIPSCHITZ condition (1. 1) on each of the subintervals of $a \leqslant x \leqslant b$ on which they are continuous ("functions of class $J(a, b)$"). It follows from results by NIKOLSKY [7] that for any such function $f(x)$ there still are a constant D_1 and a sequence of polynomials $p_n(x)$ of degree n such that

$$\int_a^b |f(x) - p_n(x)|\, dx < D_1/n, \qquad (n = 1, 2, \ldots). \tag{1. 2}$$

We shall prove that this sequence of polynomials $p_n(x) = \sum c_{nk} x^k$ can be chosen in such a way that moreover

How does this compare to known remainder theorems?

The above results are used to obtain a best possible estimate of the remainder in LITTLEWOOD's TAUBERian theorem [5] for power series. Let

$$|na_n| < K_1, \qquad (n = 1, 2, \ldots), \tag{1.5}$$

and let $\sum a_n x^n \to s$ as $x \uparrow 1$. Then LITTLEWOOD's theorem asserts that $s_n = a_0 + a_1 + \ldots + a_n \to s$ as $n \to \infty$. What can we say about the order of $|s - s_n|$ if something is known about $|\sum a_n x^n - s|$ on $0 < x < 1$? To take a simple case, assume that

$$|\sum a_n x^n - s| < K_2(1 - x), \qquad (0 < x < 1). \tag{1.6}$$

Using the above approximation theory for the case $m = 0$ it is shown that (1. 5) and (1. 6) together imply that there is a constant C such that

$$|s - s_n| < C/\log(n + 2), \qquad (n = 0, 1, \ldots), \tag{1.7}$$

where C depends only on K_1 and K_2. From the theory for $m = 1$ it follows that

$$|s - (s_0 + s_1 + \ldots + s_n)/(n + 1)| < C_1/\{\log(n + 2)\}^2, \quad (n = 0, 1, \ldots), \tag{1.8}$$

$C_1 = C_1(K_1, K_2)$, etc. The estimates (1. 7) and (1. 8) etc. are best possible (see [4] and section 5). They improve earlier results by POSTNIKOV (who proved $|s - s_n| < C(\log n)^{-\frac{1}{2}}$ for $n > n_0$, see [8]) and the author [4]. Using the methods of the present paper it can be shown that for a fairly extensive class of functions $\omega(u)$ which $\downarrow 0$ as $u \downarrow 0$ the hypotheses (1. 5) and

$$|\sum a_n x^n - s| < \omega(1 - x) \qquad (0 < x < 1) \tag{1.9}$$

imply

$$|s - s_n| < C/|\log \omega(1/n)| \qquad (n > n_0). \tag{1.10}$$

A special case of our analysis

Suppose that

$$|F(x) - s| \leq K(1-x)$$

for some constant K.

Then the correponding rate of Cauchy convergence for $F(x) \to s$ is

$$\phi(\varepsilon) = \left\lceil \frac{2K}{\varepsilon} \right\rceil$$

Plugging this in to our corollary yields the following rate of Cauchy convergence for the partial sums $\{s_n\}$:

$$\psi(\varepsilon) := LD_1^{C/\varepsilon}$$

for suitable D_1.

Rearranging, we can show that

$$|s_N - s| \leq \frac{C_1}{\log(N)}$$

for suitable C_1.

So far we have

- A quantitative analysis of Abel and Tauber's theorems (not deep, but maybe useful and a nice simple example of finitization).
- A quantitative analysis of Littlewood's Tauberian theorem (more interesting, work in progress...)

Open questions:

1. Can we finitize the Hardy-Littlewood theorem?
2. What about deeper results in Tauberian theory?
3. Extracted numerical data matches well with known results. Can we produce new "remainder theorems" which don't have any precedent in the literature?
4. Are there abstract proof theoretic metatheorems which describe and generalise certain phenomena is Tauberian theory?

THANK YOU!