A new application of proof mining in the fixed point theory of uniformly convex Banach spaces

Thomas Powell

TU Darmstadt

Logic Colloquium 2019
Special Session on Proof Theory and Proof Complexity

Prague

16 August 2019

Outline

This is a talk on **proof mining** in fixpoint theory.

I will outline results presented in the recent paper:

Powell, T. (2019). A new metastable convergence criterion and an application in the theory of uniformly convex banach spaces.
Journal of Mathematical Analysis and Applications, 478:790–805

I'll try to make everything as accessible to the non-proof miner as possible!

I intend to cover the following things:

1. Relevant background in fixpoint theory (but no introduction to proof mining!)
2. Informal overview of the proof-theoretic analysis
3. Statement of the main quantitative results

Most of the details of are **not important** for understanding the talk.

Outline

1. Background: Functional analysis
2. Conclusion: $(T^n x)$ converges
3. Assumption 1: X uniformly convex
4. Assumption 2: $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$
5. Assumption 3: $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \mathrm{Fix}(T)$
6. The main result and its corollaries
7. References

Banach's fixed point theorem

Let (X, d) be a complete metric space and $C \subseteq X$ a closed subset of X. A mapping $T : C \to C$ is a contraction if there exists some $0 \leq q < 1$ such that

$$d(Tx, Ty) \leq q \cdot d(x, y).$$

for all $x, y \in C$. The following is a classic result in metric fixed point theory.

Theorem (Banach, 1922)

If T is a contraction, then its Picard iterates $(T^n x)_{n \in \mathbb{N}}$ converge to a fixpoint of T.

This theorem no longer holds if we weaken the premise by allowing T to be nonexpansive i.e.

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in C$. E.g. For $X = \mathbb{R}$, $C = [0, 1]$ and $Tx = 1 - x$ we have

$$(T^n 0)_{n \in \mathbb{N}} = (0, 1, 0, 1, 0, 1, \ldots)$$

Picard iterates of nonexpansive maps

A natural question is the following: Under what additional conditions can we ensure that the Picard iterates $(T^n x)_{n \in \mathbb{N}}$ converges for *nonexpansive* T. For Hilbert spaces, a nonempty interior condition sufficies.

Theorem (Moreau)

Let X be a Hilbert space, $C \subseteq X$ closed and $T : C \to C$ nonexpansive. If the fixed point set $\mathrm{Fix}(T)$ *has nonempty interior, then the Picard iterates converge to a point of* $\mathrm{Fix}(T)$.

This result even holds in uniformly convex Banach spaces.

Theorem (Beauzamy)

Let X be a uniformly convex Banach space, $C \subseteq X$ closed and $T : C \to C$ nonexpansive. If the fixed point set $\mathrm{Fix}(T)$ *has nonempty interior, then the Picard iterates converge to a point of* $\mathrm{Fix}(T)$.

Uniform convexity

A Banach space is uniformly convex if for any $0 < \varepsilon \leq 2$ there is some $\delta > 0$ such that for any $\|x\| = \|y\| = 1$,

$$\frac{1}{2}\|x + y\| \geq 1 - \delta \Rightarrow \|x - y\| \leq \varepsilon$$

Intuitively: the center of a line segment inside the unit ball must lie deep inside the unit ball unless the segment is short.

Examples of uniformly convex spaces include

- all Hilbert spaces
- L^p spaces for $1 < p < \infty$

How uniform convexity is used in this talk

Let $B_r[x]$ denote the closed ball of radius r centred around r, and $B_r^o[x]$ the corresponding open ball.

The following crucial fact was discovered independently by Edelstein and Steckin:

Lemma

Suppose that X is uniformly convex. Then for any $d > 0$ and $c, c' \in X$ satisfying $0 < \|c - c'\| = hd < d$ where $0 < h < 1$ we have

$$\lim_{\delta \to \infty} \operatorname{diam}(B_{d-hd+\delta}[c] \cap (X \backslash B_d^o[c'])) = 0$$

where the convergence is uniform in c, c'.

A result of Kirk and Sims

We carry out a quantitative analysis of the following fixed point theorem of Kirk and Sims.

Theorem ([Kirk and Sims, 1999])

Suppose that C is a closed subset of a uniformly convex Banach space and $T : C \to C$ is a continuous mapping with $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$, which satisfies the condition

$$\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$$

for all $q \in \mathrm{Fix}(T)$. Then for each $x \in C$, the Picard iterates $(T^n x)_{n\in\mathbb{N}}$ converge to a fixed point of T.

Instances of Theorem KS

The condition

$$\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$$

for $q \in \mathrm{Fix}(T)$ is satisfied in particular when

- T is nonexpansive
- T is asymptotically nonexpansive, which means there exists some $\mu_n \to 1$ such that

  $$\|T^n x - T^n y\| \leq \mu_n \|x - y\|$$

  for all $x, y \in C$.

Thus, Theorem KS constitutes a very general fixed point theorem in the context of uniformly convex Banach spaces.

Proof in the case T nonexpansive

Suppose that $B_r[p] \in \mathrm{Fix}(T)$ for $p \in \mathrm{Fix}(T)$ and $r > 0$, and define

$$d := \inf_{n \in \mathbb{N}} \|T^n x - p\|$$

Assume w.l.o.g. $d \geq r$, and choose $0 < h < 1$ so that $hd < r$. For each $n \in \mathbb{N}$ choose $q_n \in \mathrm{seg}[p, T^n x]$ so that

$$\|q_n - p\| = hd \quad \text{and thus} \quad \|T^n x - q_n\| = \|T^n x - p\| - hd$$

Then $\inf_{n \in \mathbb{N}} \|T^n x - q_n\| = d - hd$ and so for any $\delta > 0$ there exists some N such that

$$\|T^N x - q_N\| \leq d - hd + \delta$$

Proof cont.

But for all $i \geq N$ we have

$$\|T^i x - q_N\| \leq \|T^N x - q_N\| \leq d - hd + \delta$$

i.e. $T^i x \in B_{d-hd+\delta}[q_N]$.

Since $\|T^i x - p\| \geq d$ have $T^i x \in X \backslash B^o_d[p]$. Thus $i \geq N$ implies

$$T^i x \in B_{d-hd+\delta}[q_N] \cap (X \backslash B^o_d[p])$$

But by the Lemma, for any $\varepsilon > 0$ can find $\delta(\varepsilon) > 0$ such that

$$\operatorname{diam}(B_{d-hd+\delta(\varepsilon)}[q] \cap (X \backslash B^o_d[p]) < \varepsilon$$

for any q with $\|q - p\| = hd$. In particular, setting $q := q_N$ for $\delta(\varepsilon)$, we have $i, j \geq N$ implies

$$\|T^i x - T^j x\| \leq \operatorname{diam}(B_{d-hd+\delta(\varepsilon)}[q] \cap (X \backslash B^o_d[p]) < \varepsilon$$

and thus $(T^n x)_{n \in \mathbb{N}}$ is Cauchy.

Structure of the theorem

We are given $T : C \to C$ for $C \subseteq X$, and some $x \in C$.

Our assumptions are

- X uniformly convex
- $\text{Int}(\text{Fix}(T)) \neq \emptyset$
- $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \text{Fix}(T)$

Our conclusion is

- $(T^n x)_{n\in\mathbb{N}}$ converges.

We will now examine each of these in turn from a **quantitative** point of view.

Outline

Cauchy convergence of $(T^n x)_{n\in\mathbb{N}}$

Our aim is to produce a quantitative version of the Cauchy convergence of the Picard iterates:

$$\forall \varepsilon > 0 \exists n \forall i, j \geq n (\|T^i x - T^j x\| \leq \varepsilon)$$

Our first question: Can we hope to extract a *direct* rate of convergence i.e. a function $\phi(\varepsilon)$ such that

$$\forall \varepsilon > 0 \exists n \leq \phi(\varepsilon) \forall i, j \geq n (\|T^i x - T^j x\| \leq \varepsilon)$$

Theorem ([Neumann, 2015, Kohlenbach, 2019])

Already for $X = \mathbb{R}$ there exists a nonexpansive mapping $T : [0,1] \to [0,1]$ (which can easily be extended to one with $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$) such that $(T^n 0)_{n\in\mathbb{N}}$ has no computable rate of convergence.

A metastable formulation of convergence

The combination of negative translation and functional interpretation, when applied to the statement that $(T^n x)_{n\in\mathbb{N}}$ is Cauchy convergent, yields:

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \forall i, j \in [n, n + g(n)](\|T^i x - T^j x\| \leq \varepsilon).$$

Our aim will be to produce a rate of metastability for the Picard iterates i.e. a functional $\Omega(\varepsilon, g)$ such that

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(\varepsilon, g) \forall i, j \in [n, n + g(n)](\|T^i x - T^j x\| \leq \varepsilon).$$

In addition to ε and g, Ω will also dependent on quantitative data from each of our assumptions.

Outline

1. Background: Functional analysis
2. Conclusion: $(T^n x)$ converges
3. Assumption 1: X uniformly convex
4. Assumption 2: $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$
5. Assumption 3: $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \mathrm{Fix}(T)$
6. The main result and its corollaries
7. References

Moduli of uniform convexity

Recall the definition of uniform convexity:

$$\forall \varepsilon \in (0,2] \exists \delta > 0 \forall x, y \in B_1[0] (\tfrac{1}{2}\|x+y\| \geq 1-\delta \to \|x-y\| \leq \varepsilon).$$

This can be given a quantitative form by considering a *modulus of uniform convexity*: This is a function $\Phi : (0,2] \to (0,1]$ satisfying

$$\forall \varepsilon \in (0,2]\ \forall x, y \in B_1[0]\ \left(\tfrac{1}{2}\|x+y\| \geq 1-\Phi(\varepsilon) \to \|x-y\| \leq \varepsilon\right). \qquad (1)$$

Moduli of uniform convexity are **widely used** in proof mining, see [Kohlenbach, 2008, Chapter 17] for a more detailed discussion.

Example

For $X = L_p$ with $2 \leq p < \infty$, a modulus of uniform convexity is given by

$$\Phi(\varepsilon) := \frac{\varepsilon^p}{p 2^p}$$

A syntactical version of Edelstein/Steckin

We use uniform convexity in a very specific form:

Lemma

Suppose that X is uniformly convex. Then for any $d > 0$ and $c, c' \in X$ satisfying $0 < \|c - c'\| = hd < d$ where $0 < h < 1$ we have

$$\lim_{\delta \to \infty} \operatorname{diam}(B_{d-hd+\delta}[c] \cap (X \backslash B^o_d[c'])) = 0$$

where the convergence is uniform in c, c'.

Actually, we identify the following syntactic, normalized version of the above:

$$\forall \varepsilon > 0, \forall h \in (0, \tfrac{1}{2}) \exists \delta \forall y \in B_1[0], u \in X$$
$$(\|u - hy\| \leq 1 - h + \delta \wedge \|u\| \geq 1 \Rightarrow \|u - y\| \leq \varepsilon)$$

A computational interpretation of Edelstein/Steckin

Lemma

Suppose that $\Phi : (0, 2] \to (0, 1]$ *is a modulus of uniform convexity for X, and define the functional* $\Psi : (0, \frac{1}{2}) \times (0, 4] \to (0, 1]$ *by*

$$\Psi(h, \varepsilon) := \min\{\tfrac{\varepsilon}{2}, 2h\Phi(\tfrac{\varepsilon}{2})\}.$$

Then we have

$$\begin{aligned}&\forall \varepsilon > 0, \forall h \in (0, \tfrac{1}{2}), y \in B_1[0], u \in X \\ &(\|u - hy\| \leq 1 - h + \Psi(h, \varepsilon) \wedge \|u\| \geq 1 \Rightarrow \|u - y\| \leq \varepsilon)\end{aligned}$$

Example

For $X = L_p$ with $2 \leq p < \infty$ we would have

$$\Psi(h, \varepsilon) = \min\left\{\frac{\varepsilon}{2}, 2h\frac{\varepsilon^p}{p2^p}\right\} = \frac{h\varepsilon^p}{p2^{p-1}}$$

for $\varepsilon \in (0, 1)$.

Outline

1. Background: Functional analysis
2. Conclusion: $(T^n x)$ converges
3. Assumption 1: X uniformly convex
4. Assumption 2: $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$
5. Assumption 3: $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \mathrm{Fix}(T)$
6. The main result and its corollaries
7. References

Not all assumptions are complicated!

$\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$ if there exists some $p \in \mathrm{Fix}(T)$ and $r > 0$ such that $B^{o}_{r}[p] \subseteq \mathrm{Fix}(T)$ i.e.

$$\forall x \in X(\underbrace{\|x - p\| <_{\mathbb{R}} r}_{\Sigma^0_1} \rightarrow \underbrace{\|Tx - x\| =_{\mathbb{R}} 0}_{\Pi^0_1})$$

The above is a *universal* statement, and thus has no computational content.

To summarise, we just need $p \in \mathrm{Fix}(T)$ and $r > 0$ with $B^{o}_{r}[p] \subseteq \mathrm{Fix}(T)$.

Outline

1. Background: Functional analysis
2. Conclusion: $(T^n x)$ converges
3. Assumption 1: X uniformly convex
4. Assumption 2: $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$
5. Assumption 3: $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \mathrm{Fix}(T)$
6. The main result and its corollaries
7. References

Sequences converging to their infimum

Unlike uniform convexity, our third assumption doesn't have a direct precedent in proof mining, and neither can we (in general) give it a direct computational interpretation.

Let's forget mappings in Banach spaces for now, and just consider sequences $(x_n)_{n\in\mathbb{N}}$ of nonnegative reals.

Our first step is to establish a syntactic criterion which is equivalent to the statement

$$\lim_{n\to\infty} x_n = \inf_{n\in\mathbb{N}} x_n.$$

This statement clearly holds if $(x_n)_{n\in\mathbb{N}}$ is monotonically decreasing, but the converse is not true.

Asymptotically decreasing sequences

We will say that a sequence is *asymptotically decreasing* if it satisfies

$$\forall \varepsilon > 0, N \exists n \forall i \geq n (x_i \leq x_N + \varepsilon).$$

This generalises the notion of (x_n) being monotonically decreasing.

Theorem

Let $(x_n)_{n \in \mathbb{N}}$ *be a sequence of nonnegative reals. Then* $(x_n)_{n \in \mathbb{N}}$ *converges to its infimum iff it is asymptotically decreasing.*

Proof.

A short exercise in elementary analysis! □

A quantitative form of asymptotic decreasingness

We now consider a metastable formulation of the notion of being asymptotically decreasing

$$\forall \varepsilon > 0, N \exists n \forall i \geq n (x_i \leq x_N + \varepsilon).$$

which corresponds to its classical functional interpretation:

$$\forall \varepsilon > 0, N, g : \mathbb{N} \to \mathbb{N} \exists n \forall i \in [n, n + g(n)] (x_i \leq x_N + \varepsilon).$$

We call $\Gamma(\varepsilon, g, N)$ a metastable rate of asymptotic decreasingness if

$$\forall \varepsilon > 0, N, g : \mathbb{N} \to \mathbb{N} \exists n \leq \Gamma(\varepsilon, g, N) \forall i \in [n, n + g(n)] (x_i \leq x_N + \varepsilon).$$

For $(x_n)_{n \in \mathbb{N}}$ decreasing, $\Gamma(\varepsilon, g, N) := N$ works.

An interesting aside

We claimed that

$$(x_n)_{n\in\mathbb{N}} \text{ asymptotically decreasing} \Rightarrow (x_n)_{n\in\mathbb{N}} \text{ converges}$$

The following is a computational interpretation of this result.

Theorem (P.)

Let $(x_n)_{n\in\mathbb{N}}$ be a sequence of nonnegative reals bounded above by K, with a metastable rate of asymptotic decreasingness Γ. Define

$$\Omega(\varepsilon, g) := \Gamma^*(\tfrac{\varepsilon}{2}, g, f^{\lceil 2K/\varepsilon \rceil}(0))$$

where

- $f(j) := \Gamma^*(\frac{\varepsilon}{2}, g, j) + g^*(\Gamma^*(\frac{\varepsilon}{2}, g, j))$
- $g^*(j) := \max_{i\le j}\{j, g(i)\}$ *and* $\Gamma^*(\frac{\varepsilon}{2}, g, j) := \max_{i\le j}\{j, \Gamma(\frac{\varepsilon}{2}, g, i)\}$.

Then Ω is a rate of metastability for $(x_n)_{n\in\mathbb{N}}$ i.e.

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \le \Omega(\varepsilon, g) \forall i, j \in [n, n + g(n)](|x_i - x_j| \le \varepsilon).$$

A quantitative version of our third assumption

We are now ready to give a quantitative formulation to the statement

$$\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\| \quad \text{for all } q \in \mathrm{Fix}(T)$$

In addition, we now demand a *uniformity* assumption.

Let $p \in X$ and $r > 0$ with $B_r[p] \in \mathrm{Fix}(T)$, and suppose that $\|x - p\| \leq K$.
We call $\Gamma(K, r, \varepsilon, g, N)$ a *uniform* metastable rate of asymptotic decreasingness for $(\|T^n x - q\|)_{n\in\mathbb{N}}$ if

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N}, N \exists n \leq \Gamma(K, r, \varepsilon, g, N) \forall i \in [n, n + g(n)]$$
$$(\|T^i x - q\| \leq \|T^N x - q\| + \varepsilon)$$

for all $q \in B_r[p]$.

Outline

1. Background: Functional analysis
2. Conclusion: $(T^n x)$ converges
3. Assumption 1: X uniformly convex
4. Assumption 2: $\mathrm{Int}(\mathrm{Fix}(T)) \neq \emptyset$
5. Assumption 3: $\lim_{n\to\infty} \|T^n x - q\| = \inf_{n\in\mathbb{N}} \|T^n x - q\|$ for all $q \in \mathrm{Fix}(T)$
6. The main result and its corollaries
7. References

Main result

Theorem (P.)

Let X be a Banach space, $T : C \to C$ a mapping and $x \in C$. Suppose that

- $B_r[p] \subseteq \mathrm{Fix}(T)$ *for* $p \in X$ *with* $\|x - p\| \leq K$ *and* $r > 0$;
- Φ *is a modulus of uniform convexity for* X;
- Γ *is a metastable rate of asymptotic decreasingness for* $(\|T^n x - q\|)_{n\in\mathbb{N}}$ *uniform for* $q \in B_r[p]$.

Then

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(\Phi, \Gamma, K, r, \varepsilon, g) \forall i, j \in [n, n + g(n)](\|T^i x - T^j x\| \leq \varepsilon)$$

for Ω *defined as follows:*

- $\Omega(\Phi, \Gamma, K, r, \varepsilon, g) := \Gamma^*(K, r, \eta, g, f^{(\lceil K/\eta \rceil)}(0))$;
- $f(j) := \Gamma^*(K, r, \eta, g, j) + g^*(\Gamma^*(K, r, \eta, g, j))$;
- $\eta := \frac{r}{4} \cdot \min\{1, \Psi(\min\{\frac{1}{4}, \frac{r}{K}\}, \frac{\varepsilon}{2K})\}$;
- $\Psi(h, \varepsilon') := \min\{\frac{\varepsilon'}{2}, 2h\Phi(\frac{\varepsilon'}{2})\}$.

In the case where T is nonexpansive

Corollary (P.)

Let X be a Banach space, $T : C \to C$ a nonexpansive mapping and $x \in C$. Suppose that

- $B_r[p] \subseteq \mathrm{Fix}(T)$ *for* $p \in X$ *with* $\|x - p\| \leq K$ *and* $r > 0$;
- Φ *is a modulus of uniform convexity for* X;

Then

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(\Phi, \Gamma, K, r, \varepsilon, g) \forall i, j \in [n, n + g(n)] (\|T^i x - T^j x\| \leq \varepsilon)$$

for Ω *defined as follows:*

- $\Omega(\Phi, K, r, \varepsilon, g) := f^{(\lceil K/\eta \rceil)}(0)$;
- $f(j) := j + g^*(j)$;
- $\eta := \frac{r}{4} \cdot \min\{1, \Psi(\min\{\frac{1}{4}, \frac{r}{K}\}, \frac{\varepsilon}{2K})\}$;
- $\Psi(h, \varepsilon') := \min\{\frac{\varepsilon'}{2}, 2h\Phi(\frac{\varepsilon'}{2})\}$.

Asymptotic regularity of the Picard iterates

If the Picard iterates converge, then in particular, they must be asymptotically regular:

$$\forall \varepsilon > 0 \exists n \forall i \geq n(\|T^{i+1}x - T^i x\| \leq \varepsilon).$$

In the case that T is nonexpansive, asymptotic regularity is equivalent to the following $\forall\exists$ statement:

$$\forall \varepsilon > 0 \exists n \left(\|T^{n+1}x - T^n x\| \leq \varepsilon\right).$$

This would suggest it is possible to extract a *direct* rate of asymptotic regularity in our setting i.e. a function $f(\varepsilon)$ such that

$$\forall \varepsilon > 0, i \geq f(\varepsilon)(\|T^{i+1}x - T^i x\| \leq \varepsilon).$$

A rate of asymptotic regularity

Theorem (P.)

Let X be a Banach space, $T : C \to C$ a nonexpansive mapping and $x \in C$. Suppose that

- $B_r[p] \subseteq \mathrm{Fix}(T)$ *for* $p \in X$ *with* $\|x - p\| \leq K$ *and* $r > 0$;
- Φ *is a modulus of uniform convexity for* X;

Then

$$\forall \varepsilon > 0, i \geq f(\varepsilon)(\|T^{i+1}x - T^i x\| \leq \varepsilon)$$

where

- $f(\varepsilon) := \lceil K/\eta \rceil$;
- $\eta := \frac{r}{4} \cdot \min\{1, \Psi(\min\{\frac{1}{4}, \frac{r}{K}\}, \frac{\varepsilon}{2K})\}$;
- $\Psi(h, \varepsilon') := \min\{\frac{\varepsilon'}{2}, 2h\Phi(\frac{\varepsilon'}{2})\}$.

A concrete result for L^p spaces

Theorem (P.)

Let $T : C \to C$ be a nonexpansive mapping for $C \subseteq L^p$ and $x \in C$. Suppose that $B_r[p'] \subseteq \mathrm{Fix}(T)$ for $p' \in X$ with $\|x - p'\| \leq K$ and $r > 0$. Then

$$\forall \varepsilon > 0, i \geq f(\varepsilon)(\|T^{i+1}x - T^i x\| \leq \varepsilon)$$

where

$$f(\varepsilon) := \left\lceil \frac{p \cdot 2^{3p+1} \cdot K^{p+2}}{\varepsilon^p \cdot r^2} \right\rceil$$

Note that this is a *purely mathematical* result. There is no mention of proof interpretations, higher-order functionals, metastability etc.

Lemma (P.)

Suppose that $T : C \to C$ *is asymptotically nonexpansive i.e. there exists some* $\mu_n \to 1$ *such that*

$$\|T^n x - T^n y\| \leq \mu_n \|x - y\|$$

for all $x, y \in C$.

Suppose, moreover, that $(\mu_n)_{n \in \mathbb{N}}$ *is decreasing and has a rate of convergence* $c(\delta)$ *i.e.*

$$\forall \delta > 0 (\mu_{c(\delta)} \leq 1 + \delta)$$

Then

$$\Gamma(K, r, \varepsilon, N) := N + c\left(\frac{\varepsilon}{\mu_0 (K + r)}\right)$$

is a direct rate of asymptotic decreasingness for $(\|T^n x - q\|)_{n \in \mathbb{N}}$ *uniform for* $q \in B_r[p]$, *where* $\|x - p\| \leq K$.

One final corollary

Theorem (P.)

Let X be a Banach space, $T : C \to C$ an asymptotically nonexpansive mapping relative to some decreasing $(\mu_n)_{n\in\mathbb{N}}$, and $x \in C$. Suppose that

- *c is a rate of convergence of $(\mu_n)_{n\in\mathbb{N}}$*
- *$B_r[p] \subseteq \mathrm{Fix}(T)$ for $p \in X$ with $\|x - p\| \leq K$ and $r > 0$;*
- *Φ is a modulus of uniform convexity for X;*

Then

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists n \leq \Omega(c, \Gamma, K, r, \varepsilon, g) \forall i, j \in [n, n + g(n)] (\|T^i x - T^j x\| \leq \varepsilon)$$

for Ω defined as follows:

- $\Omega(c, \Phi, K, r, \varepsilon, g) := (f_\omega)^{(\lceil K/\eta \rceil)}(0) + \omega;$
- $f_\omega(j) := j + \omega + g^*(j + \omega);$
- $\omega := c\left(\frac{\eta}{\mu_0(K+r)}\right);$
- $\eta := \frac{r}{4} \cdot \min\{1, \Psi(\min\{\frac{1}{4}, \frac{r}{K}\}, \frac{\varepsilon}{2K})\};$
- $\Psi(h, \varepsilon') := \min\{\frac{\varepsilon'}{2}, 2h\Phi(\frac{\varepsilon'}{2})\}.$

Outline

Thank you!

Kirk, W. A. and Sims, B. (1999).
Convergence of Picard iterates of nonexpansive mappings.
Bulletin of the Polish Academy of Sciences, (47):147–155.

Kohlenbach, U. (2008).
Applied Proof Theory: Proof Interpretations and their Use in Mathematics.
Monographs in Mathematics. Springer.

Kohlenbach, U. (2019).
On the reverse mathematics and Weihrauch complexity of moduli of regularity and uniqueness.
To appear in Computability.

Neumann, E. (2015).
Computational problems in metrix fixed point theory and their Weihrauch degrees.
Logical Methods in Computer Science, 11:1–44.

Powell, T. (2019).
A new metastable convergence criterion and an application in the theory of uniformly convex banach spaces.
Journal of Mathematical Analysis and Applications, 478:790–805.