Variations on learning: Relating the epsilon calculus to proof interpretations

Thomas Powell

University of Innsbruck

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University of Montpellier
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Background

APPLIED PROOF THEORY $\approx$ PROOF INTERPRETATIONS

- Heavy use and development of negative translations, Dialectica interpretation, modified realizability and so on...
- Main purpose is to extract computational content from classical proofs.
- Applications in logic (e.g. characterisation of prov. rec. functions), mathematics (see proof mining program), theoretical computer science (e.g. complexity analysis of programs).

Background

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Epsilon calculus and elimination procedure

Initially used by Kreisel in his ‘unwinding’ program (1951), later replaced by proof interpretations... now rarely seen in this context. But offers an elegant interpretation of non-constructive reasoning via explicit backtracking/learning - a feature subtly hidden in realizing terms extracted via proof interpretations.

Theorem (Drinker's paradox). $\exists n \forall m (P(m) \to P(n))$. *"In a pub there is always a person, such that if anyone is drinking, then that person is drinking"*

Derivation in classical predicate logic (assume P decidable, 0 some canonical element):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{P(k) \to \forall m (P(m) \to P(k))}\ \forall\mathrm{r}}{P(k) \to \exists n \forall m (P(m) \to P(n))}\ \exists\mathrm{ax}}{\exists k P(k) \to \exists n \forall m (P(m) \to P(n))}\ \exists\mathrm{r}
\qquad
\dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\forall k \neg P(k) \to P(m) \to P(0)}\ \forall\mathrm{ax}}{\forall k \neg P(k) \to \forall m (P(m) \to P(0))}\ \forall\mathrm{r}}{\forall k \neg P(k) \to \exists n \forall m (P(m) \to P(n))}\ \exists\mathrm{ax}}{\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m (P(m) \to P(n)) \vee \exists n \forall m (P(m) \to P(n))}}{\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m (P(m) \to P(n))}\ \mathrm{ctr}}{\exists n \forall m (P(m) \to P(n))}\ \mathsf{LEM}
$$

Question

What is the constructive meaning of the Drinker's paradox?

Method based on ε-elimination

STEP 1. Replace the quantifiers with non-constructive ϵ-terms:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{P(k) \to \forall m(P(m) \to P(k))}\ \forall\mathrm{r}}{P(k) \to \exists n \forall m(P(m) \to P(n))}\ \exists\mathrm{ax}}{\exists k P(k) \to \exists n \forall m(P(m) \to P(n))}\ \exists\mathrm{r} \qquad \dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\forall k \neg P(k) \to P(m) \to P(0)}\ \forall\mathrm{ax}}{\forall k \neg P(k) \to \forall m(P(m) \to P(0))}\ \forall\mathrm{r}}{\forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n))}\ \exists\mathrm{ax}}{\dfrac{\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n)) \vee \exists n \forall m(P(m) \to P(n))}{\dfrac{\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n))}{\exists n \forall m(P(m) \to P(n))}\ \mathsf{LEM}}\ \mathrm{ctr}}
$$

Critical formulas

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-
-

Method based on ε-elimination

STEP 1. Replace the quantifiers with non-constructive ϵ-terms:

$$
\dfrac{
\dfrac{\dfrac{\dfrac{P(\epsilon_k) \to P(m) \to P(\epsilon_k)}{P(\epsilon_k) \to \forall m(P(m) \to P(\epsilon_k))}\,\forall\mathrm{r}}{P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n))}\,\exists\mathrm{ax}}{P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n))}\,\exists\mathrm{r}
\qquad
\dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\neg P(\epsilon_k) \to P(m) \to P(0)}\,\boxed{\forall\mathrm{ax}}}{\neg P(\epsilon_k) \to \forall m(P(m) \to P(0))}\,\forall\mathrm{r}}{\neg P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n))}\,\exists\mathrm{ax}
}{
\dfrac{\dfrac{P(\epsilon_k) \vee \neg P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n)) \vee \exists n \forall m(P(m) \to P(n))}{\boxed{P(\epsilon_k) \vee \neg P(\epsilon_k)} \to \exists n \forall m(P(m) \to P(n))}\,\mathrm{ctr}}{\exists n \forall m(P(m) \to P(n))}
}
$$

Critical formulas

- $P(m) \to P(\epsilon_k)$
-
-

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$$
\dfrac{
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\dfrac{
\dfrac{P(\epsilon_k) \to P(m) \to P(\epsilon_k)}{P(\epsilon_k) \to \forall m(P(m) \to P(\epsilon_k))}\,\forall\mathrm{r}
}{P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n))}\,\exists\mathrm{ax}
\qquad
\dfrac{
\dfrac{
\dfrac{\neg P(m) \to P(m) \to P(0)}{\neg P(\epsilon_k) \to P(m) \to P(0)}\,\boxed{\forall\mathrm{ax}}
}{\neg P(\epsilon_k) \to \forall m(P(m) \to P(0))}\,\forall\mathrm{r}
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}{P(\epsilon_k) \vee \neg P(\epsilon_k) \to \exists n \forall m(P(m) \to P(n)) \vee \exists n \forall m(P(m) \to P(n))}\,\mathrm{ctr}
}{
\dfrac{\boxed{P(\epsilon_k) \vee \neg P(\epsilon_k)} \to \exists n \forall m(P(m) \to P(n))}{\exists n \forall m(P(m) \to P(n))}
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\dfrac{
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\dfrac{P(\epsilon_k) \to P(\epsilon_m \epsilon_k) \to P(\epsilon_k)}{P(\epsilon_k) \to P(\epsilon_m \epsilon_k) \to P(\epsilon_k)}\;\forall\mathrm{r}
}{P(\epsilon_k) \to \exists n (P(\epsilon_m n) \to P(n))}\;\exists\mathrm{ax}
\qquad
\dfrac{
\dfrac{
\dfrac{\neg P(\epsilon_m 0) \to P(\epsilon_m 0) \to P(0)}{\neg P(\epsilon_k) \to P(\epsilon_m 0) \to P(0)}\;\boxed{\forall\mathrm{ax}}
}{\neg P(\epsilon_k) \to P(\epsilon_m 0) \to P(0)}\;\forall\mathrm{r}
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}{
P(\epsilon_k) \vee \neg P(\epsilon_k) \to \exists n (P(\epsilon_m n) \to P(n)) \vee \exists n (P(\epsilon_m n) \to P(n))
}
}{
\dfrac{P(\epsilon_k) \vee \neg P(\epsilon_k) \to \exists n (P(\epsilon_m n) \to P(n))}{\exists n (P(\epsilon_m n) \to P(n))}
}\;\mathrm{ctr}
$$

Critical formulas

- $P(\epsilon_m 0) \to P(\epsilon_k)$
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Critical formulas

- $P(\epsilon_m 0) \to P(\epsilon_k)$
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Method based on ϵ-elimination

STEP 2. Solve the set of critical formulas using an update procedure:

Critical formulas

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Solution

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Solution

Try $\epsilon_k = \epsilon_n = 0$...

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Solution

Try $\epsilon_k = \epsilon_n = 0$...

- $P(\epsilon_m 0) \to P(0)$
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- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m \epsilon_n) \to P(\epsilon_n))$

Solution

Try $\epsilon_k = \epsilon_n = 0$... Works unless $P(\epsilon_m 0) \wedge \neg P(0)$.

- $P(\epsilon_m 0) \to P(0)$
- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m 0) \to P(0))$
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- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m \epsilon_n) \to P(\epsilon_n))$

Solution

Assuming $P(\epsilon_m 0) \wedge \neg P(0)$, try $\epsilon_k = \epsilon_m 0$, $\epsilon_n = 0$.

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-
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Method based on ϵ-elimination

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- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m \epsilon_n) \to P(\epsilon_n))$

Solution

Assuming $P(\epsilon_m 0) \wedge \neg P(0)$, try $\epsilon_k = \epsilon_m 0$, $\epsilon_n = 0$. This fails...

- $P(\epsilon_m 0) \to P(\epsilon_m 0)$
- $(P(\epsilon_m \epsilon_m 0) \to P(\epsilon_m 0)) \to (P(\epsilon_m 0) \to P(0))$
- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m 0) \to P(0))$

Method based on ϵ-elimination

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- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m \epsilon_n) \to P(\epsilon_n))$

Solution

Assuming $P(\epsilon_m 0) \wedge \neg P(0)$, try $\epsilon_k = \epsilon_n = \epsilon_m 0$.

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Solution

Assuming $P(\epsilon_m 0) \wedge \neg P(0)$, try $\epsilon_k = \epsilon_n = \epsilon_m 0$. Works.

- $P(\epsilon_m 0) \to P(\epsilon_m 0)$
- $(P(\epsilon_m \epsilon_m 0) \to P(\epsilon_m 0)) \to (P(\epsilon_m \epsilon_m 0) \to P(\epsilon_m 0))$
- $(P(\epsilon_m 0) \to P(0)) \to (P(\epsilon_m \epsilon_m 0) \to P(\epsilon_m 0))$

Method based on ϵ-elimination

NON-CONSTRUCTIVE THEOREM: $\exists n \forall m (P(m) \to P(n))$.

CONSTRUCTIVE FINITIZATION: For an arbitrary $\epsilon_m \colon \mathbb{N} \to \mathbb{N}$ there exists some $\epsilon_n \colon \mathbb{N}$ satisfying

$$P(\epsilon_m \epsilon_n) \to P(\epsilon_n),$$

which can be computed by the algorithm

$$\epsilon_{\mathtt{n}} := \begin{cases} \mathtt{0} & \mathtt{if}\ \mathtt{P}(\epsilon_{\mathtt{m}}\mathtt{0}) \to \mathtt{P}(\mathtt{0}) \\ \epsilon_{\mathtt{m}}\mathtt{0} & \mathtt{otherwise} \end{cases}$$

Method based on ϵ-elimination

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Remark. The term ϵ_m forms a measure of how we might *use* the theorem as a lemma in a proof i.e. exactly when we need the $\forall$-axiom

$$\exists n \forall m (P(m) \to P(n)) \to \exists n (A(t) \to A(n)).$$

The Dialectica interpretation in one slide!

- Invented by Gödel in the 1930s, published in famous 1958 paper *Über eine bisher noch nicht benützte Erweiterung des finiten Standpunktes*
- Converts formulas in language of predicate logic to a special kind of Skolem form $A \mapsto \exists \vec{x} \forall \vec{y} A^D(\vec{x}, \vec{y})$.
- Provides an algorithm for extracting realizers for $\exists \vec{x}$ from intuitionistic proof of A by interpreting all axioms and rules of theory. Only non-trivial element is contraction, which is interpreted using case-definition and characteristic functions for quantifier-free formulas.
- For classical logic, Dialectica interpretation is precomposed by a negative translation. For Σ_2-formulas this yields the 'no-counterexample interpretation' of Kreisel:

$$\begin{aligned} \exists x^{\mathbb{N}} \forall y^{\mathbb{N}} Q(x,y) &\mapsto \neg\neg \exists x \forall y Q(x,y) \\ &\mapsto \exists X^{(\mathbb{N}\to\mathbb{N})\to\mathbb{N}} \forall f^{\mathbb{N}\to\mathbb{N}} Q(Xf, f(Xf)). \end{aligned}$$

Method based on Dialectica interpretation

STEP 1. Translate the proof to obtain a derivation of the *negative translation* of the theorem:

$$
\begin{prooftree}
\AxiomC{$P(k) \to P(m) \to P(k)$}
\RightLabel{$\forall$r}
\UnaryInfC{$P(k) \to \forall m(P(m) \to P(k))$}
\RightLabel{$\exists$ax}
\UnaryInfC{$P(k) \to \exists n \forall m(P(m) \to P(n))$}
\RightLabel{$\exists$r}
\UnaryInfC{$\exists k P(k) \to \exists n \forall m(P(m) \to P(n))$}
\AxiomC{$\neg P(m) \to P(m) \to P(0)$}
\RightLabel{$\forall$ax}
\UnaryInfC{$\forall k \neg P(k) \to P(m) \to P(0)$}
\RightLabel{$\forall$r}
\UnaryInfC{$\forall k \neg P(k) \to \forall m(P(m) \to P(0))$}
\RightLabel{$\exists$ax}
\UnaryInfC{$\forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n))$}
\BinaryInfC{$\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n)) \vee \exists n \forall m(P(m) \to P(n))$}
\RightLabel{ctr}
\UnaryInfC{$\exists k P(k) \vee \forall k \neg P(k) \to \exists n \forall m(P(m) \to P(n))$}
\RightLabel{LEM}
\UnaryInfC{$\exists n \forall m(P(m) \to P(n))$}
\end{prooftree}
$$

Method based on Dialectica interpretation

STEP 1. Translate the proof to obtain a derivation of the *negative translation* of the theorem:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{P(k) \to \forall m(P(m) \to P(k))}}{P(k) \to \neg\neg\exists n \forall m(P(m) \to P(n))}}{\exists k P(k) \to \neg\neg\exists n \forall m(P(m) \to P(n))} \qquad \dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\forall k \neg P(k) \to P(m) \to P(0)}}{\forall k \neg P(k) \to \forall m(P(m) \to P(0))}}{\forall k \neg P(k) \to \neg\neg\exists n \forall m(P(m) \to P(n))}}{\exists k P(k) \vee \forall k \neg P(k) \to \neg\neg\exists n \forall m(P(m) \to P(n)) \vee \neg\neg\exists n \forall m(P(m) \to P(n))}\,\text{ctr}}{\exists k P(k) \vee \forall k \neg P(k) \to \neg\neg\exists n \forall m(P(m) \to P(n))}\,\mathsf{LEM}}{\neg\neg\exists n \forall m(P(m) \to P(n))}$$

Method based on Dialectica interpretation

STEP 2. Eliminate double negations by creating quantifier dependencies (cf. no-counterexample interpretation):

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{P(k) \to \forall m (P(m) \to P(k))}}{P(k) \to \exists N \forall f (P(f(Nf)) \to P(Nf))}}{\exists k P(k) \to \exists N \forall f (P(f(Nf)) \to P(Nf))} \qquad \dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\forall k \neg P(k) \to P(m) \to P(0)}}{\forall k \neg P(k) \to \forall m (P(m) \to P(0))}}{\forall k \neg P(k) \to \exists N \forall f (P(f(Nf)) \to P(Nf))}}{\exists k P(k) \vee \forall k \neg P(k) \to \exists N \forall f (P(f(Nf)) \to P(Nf)) \vee \exists N \forall f (P(f(Nf)) \to P(Nf))}\;\text{ctr}}{\exists k P(k) \vee \forall k \neg P(k) \to \exists N \forall f (P(f(Nf)) \to P(Nf))}\;\mathsf{LEM}}{\exists N \forall f (P(f(Nf)) \to P(Nf))}
$$

Method based on Dialectica interpretation

STEP 3. Skolemize each sentence:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(k) \to P(m) \to P(k)}{\forall m[P(k) \to P(m) \to P(k)]}}{\exists N \forall f[P(k) \to P(f(Nf)) \to P(Nf)]}}{\exists N \forall f, k[P(k) \to P(f(Nkf)) \to P(Nkf)]} \qquad \cfrac{\cfrac{\cfrac{\neg P(m) \to P(m) \to P(0)}{\exists k[\neg P(k) \to P(m) \to P(0)]}}{\exists k \forall m[\neg P(k) \to P(m) \to P(0)]}}{\exists K, N \forall f[\neg P(Kf) \to P(f(Nf)) \to P(Nf)]}}{\cfrac{\exists K, N, N' \forall k, f, f'[P(k) \vee \neg P(K_{kff'}) \to (P(f(N_{kf})) \to P(N_{kf})) \vee (P(f'(N'_{kf'})) \to P(N'_{kf'}))]}{\cfrac{\exists K, N \forall k, f[P(k) \vee \neg P(K_{kf}) \to (P(f(N_{kf})) \to P(N_{kf}))]}{\exists N \forall f[P(f(Nf)) \to P(Nf)]}\ \text{LEM}}}$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{
\dfrac{P(k) \to P(m) \to P(k)}{\forall m[P(k) \to P(m) \to P(k)]}
}{\exists N \forall f[P(k) \to P(f(Nf)) \to P(Nf)]}
}{\exists N \forall f, k[P(k) \to P(f(Nkf)) \to P(Nkf)]}
\qquad
\dfrac{
\dfrac{
\dfrac{\neg P(m) \to P(m) \to P(0)}{\exists k[\neg P(k) \to P(m) \to P(0)]}
}{\exists k \forall m[\neg P(k) \to P(m) \to P(0)]}
}{\exists K, N \forall f[\neg P(Kf) \to P(f(Nf)) \to P(Nf)]}
}{\exists K, N, N' \forall k, f, f'[P(k) \vee \neg P(K_{kff'}) \to (P(f(N_{kf})) \to P(N_{kf})) \vee (P(f'(N'_{kf'})) \to P(N'_{kf'}))]}
}{\exists K, N \forall k, f[P(k) \vee \neg P(K_{kf}) \to (P(f(N_{kf})) \to P(N_{kf}))]}
}{\exists N \forall f[P(f(Nf)) \to P(Nf)]}\ \text{LEM}
$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{\forall m[P(k) \to P(m) \to P(k)]}}{\forall f[P(k) \to P(f(k)) \to P(k)]}}{\forall f, k[P(k) \to P(f(k)) \to P(k)]} \qquad \dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{\exists k[\neg P(k) \to P(m) \to P(0)]}}{\exists k \forall m[\neg P(k) \to P(m) \to P(0)]}}{\exists K, N \forall f[\neg P(Kf) \to P(f(Nf)) \to P(Nf)]}}{\exists K, N, N' \forall k, f, f'[P(k) \vee \neg P(K_{kff'}) \to (P(f(N_{kf})) \to P(N_{kf})) \vee (P(f'(N'_{kf'})) \to P(N'_{kf'}))]}}{\exists K, N \forall k, f[P(k) \vee \neg P(K_{kf}) \to (P(f(N_{kf})) \to P(N_{kf}))]}}{\exists N \forall f[P(f(Nf)) \to P(Nf)]} \ \text{LEM}$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k)\to P(m)\to P(k)}{\forall m[P(k)\to P(m)\to P(k)]}}{\forall f[P(k)\to P(f(k))\to P(k)]}}{\forall f,k[P(k)\to P(f(k))\to P(k)]} \qquad \dfrac{\dfrac{\dfrac{\neg P(m)\to P(m)\to P(0)}{[\neg P(m)\to P(m)\to P(0)]}}{\forall m[\neg P(m)\to P(m)\to P(0)]}}{\forall f[\neg P(f0)\to P(f0)\to P(0)]}}{\exists K,N,N'\forall k,f,f'[P(k)\vee\neg P(K_{kff'})\to(P(f(N_{kf}))\to P(N_{kf}))\vee(P(f'(N'_{kf'}))\to P(N'_{kf'}))]}}{\exists K,N\forall k,f[P(k)\vee\neg P(K_{kf})\to(P(f(N_{kf}))\to P(N_{kf}))]}}{\exists N\forall f[P(f(Nf))\to P(Nf)]}\ \text{LEM}$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{\cfrac{P(k) \to P(m) \to P(k)}{\forall m[P(k) \to P(m) \to P(k)]}}{\forall f[P(k) \to P(f(k)) \to P(k)]}}{\forall f, k[P(k) \to P(f(k)) \to P(k)]} \qquad \cfrac{\cfrac{\cfrac{\neg P(m) \to P(m) \to P(0)}{[\neg P(m) \to P(m) \to P(0)]}}{\forall m[\neg P(m) \to P(m) \to P(0)]}}{\forall f[\neg P(f0) \to P(f(0)) \to P(0)]}}{\forall k, f, f'[P(k) \vee \neg P(f'(0)) \to (P(f(k)) \to P(k)) \vee (P(f'(0)) \to P(0))]}}{\exists K, N \forall k, f[P(k) \vee \neg P(K_{kf}) \to (P(f(N_{kf})) \to P(N_{kf}))]}\ \text{ctr}}{\exists N \forall f[P(f(Nf)) \to P(Nf)]}\ \mathsf{LEM}$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$
\begin{array}{c}
\begin{array}{c}
P(k) \to P(m) \to P(k) \\ \hline
\forall m[P(k) \to P(m) \to P(k)] \\ \hline
\forall f[P(k) \to P(f(k)) \to P(k)] \\ \hline
\forall f, k[P(k) \to P(f(k)) \to P(k)]
\end{array}
\qquad
\begin{array}{c}
\neg P(m) \to P(m) \to P(0) \\ \hline
[\neg P(m) \to P(m) \to P(0)] \\ \hline
\forall m[\neg P(m) \to P(m) \to P(0)] \\ \hline
\forall f[\neg P(f0) \to P(f(0)) \to P(0)]
\end{array} \\ \hline
\forall k, f, f'[P(k) \vee \neg P(f(0)) \to (P(f(k)) \to P(k)) \vee (P(f'(0)) \to P(0))] \\ \hline
\forall k, f[P(k) \vee \neg P(f(0)) \to (P(f(N_{kf})) \to P(N_{kf}))] \\ \hline
\exists N \forall f[P(f(Nf)) \to P(Nf)]
\end{array}
\ \text{LEM}
$$

$$
N_{kf} := \begin{cases} 0 & \text{if } P(f(0)) \to P(0) \\ k & \text{otherwise} \end{cases}
$$

Method based on Dialectica interpretation

STEP 4. Extract, over the structure of the proof, realizing terms for the existential quantifiers:

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{P(k) \to P(m) \to P(k)}{\forall m[P(k) \to P(m) \to P(k)]}}{\forall f[P(k) \to P(f(k)) \to P(k)]}}{\forall f,k[P(k) \to P(f(k)) \to P(k)]} \qquad \dfrac{\dfrac{\dfrac{\neg P(m) \to P(m) \to P(0)}{[\neg P(m) \to P(m) \to P(0)]}}{\forall m[\neg P(m) \to P(m) \to P(0)]}}{\forall f[\neg P(f0) \to P(f(0)) \to P(0)]}}{\dfrac{\dfrac{\forall k,f,f'[P(k) \vee \neg P(f(0)) \to (P(f(k)) \to P(k)) \vee (P(f'(0)) \to P(0))]}{\forall k,f[P(k) \vee \neg P(f(0)) \to (P(f(N_{kf})) \to P(N_{kf}))]}}{\forall f[P(f(0)) \vee \neg P(f(0)) \to (P(f(N_{kf})) \to P(N_{kf}))]}}}{\forall f[P(f(Nf)) \to P(Nf)]}
$$

$$
Nf := \begin{cases} 0 & \text{if } P(f(0)) \to P(0) \\ f(0) & \text{otherwise} \end{cases}
$$

Method based on Dialectica interpretation

NON-CONSTRUCTIVE THEOREM: $\exists n \forall m (P(m) \to P(n))$.

CONSTRUCTIVE FINITIZATION: For an arbitrary $f\colon \mathbb{N} \to \mathbb{N}$ there exists some $n\colon \mathbb{N}$ satisfying

$$P(f(n)) \to P(n),$$

which can be computed by the algorithm

$$\mathtt{Nf} := \begin{cases} \mathtt{0} & \mathtt{if\ P(f0)} \to \mathtt{P(0)} \\ \mathtt{f0} & \mathtt{otherwise} \end{cases}$$

Method based on Dialectica interpretation

NON-CONSTRUCTIVE THEOREM: $\exists n \forall m (P(m) \to P(n))$.

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which can be computed by the algorithm

$$\mathtt{Nf} := \begin{cases} \mathtt{0} & \mathtt{if\ P(f0)} \to \mathtt{P(0)} \\ \mathtt{f0} & \mathtt{otherwise} \end{cases}$$

Remark. The 'counterexample function' f again indicates of how we might *use* the theorem as a lemma in a proof: in such a proof it will take a set of concrete instantiations.

This looks familiar!

NON-CONSTRUCTIVE THEOREM: $\exists n \forall m (P(m) \to P(n))$.

CONSTRUCTIVE FINITIZATION: For an arbitrary $\epsilon_m \colon \mathbb{N} \to \mathbb{N}$ there exists some $\epsilon_n \colon \mathbb{N}$ satisfying

$$P(\epsilon_m \epsilon_n) \to P(\epsilon_n),$$

which can be computed by the algorithm

$$\epsilon_{\mathtt{n}} := \begin{cases} \mathtt{0} & \mathtt{if}\ \mathtt{P}(\epsilon_{\mathtt{m}}\mathtt{0}) \to \mathtt{P}(\mathtt{0}) \\ \epsilon_{\mathtt{m}}\mathtt{0} & \mathtt{otherwise} \end{cases}$$

Remark. The term ϵ_m forms a measure of how we might *use* the theorem as a lemma in a proof i.e. exactly when we need the $\forall$-axiom

$$\exists n \forall m (P(m) \to P(n)) \to \exists n (A(t) \to A(n)).$$

Summary

Epsilon calculus

- Classical logic interpreted *directly* through the elimination of quantifiers and the extraction of critical formulas.
- Realizer obtained through an *explicit* learning procedure given by epsilon substitution method.

Dialectica interpretation

- Classical logic interpreted *indirectly* through an intermediate negative translation.
- Realizing term extracted recursively from negative translated proof, and implements learning *implicitly* through the interpretation of contraction.

Questions

1. What is the relationship between programs extracted using functional interpretations like Dialectica, and those extracted using methods based explicitly on learning e.g. epsilon calculus, more modern variants such as interactive realizability?
2. Programs extracted using Dialectica are typically highly complex λ-terms in (some extension of) system T. Can we characterise their operational behaviour as some kind of intelligent learning procedure?
3. Could we obtain greater insight into how Dialectica-programs behave by reformulating the traditional soundness proofs, and replacing usual recursors used to interpret axioms and rules by new forms of recursion based explicitly on learning?

Dialectica interpretation of classical analysis

Suppose that we have a countable sequence of instances of Σ_1^0-LEM:

$$\forall i\ .\ \exists k P_i(k) \vee \forall k \neg P_i(k).$$

An instance of axiom of countable choice yields a Skolem function $\chi\colon \mathbb{N} \to \mathbb{N}$ satisfying

$$\forall i, k\ .\ P_i(\chi i) \vee \neg P_i(k).$$

This is already very strong e.g. setting

$$gi = \texttt{true} \text{ if } P_i(\chi i) \text{ else } \texttt{false}.$$

gives us a comprehension function for $\exists k P_i(k)$ satisfying

$$\forall i\ .\ gi = \texttt{true} \leftrightarrow \exists k P_i(k).$$

Dialectica interpretation of classical analysis

$$\forall i \,.\, \exists k P_i(k) \vee \forall k \neg P_i(k)$$

Dialectica interpretation of classical analysis

$$\forall i \neg\neg(\exists k P_i(k) \vee \forall k \neg P_i(k))$$

Dialectica interpretation of classical analysis

$$(*) \quad \forall i, f \exists k (P_i(k) \vee \neg P_i(f(k)))$$

Dialectica interpretation of classical analysis

$$(*) \quad \forall i, f \exists k (P_i(k) \vee \neg P_i(f(k)))$$

Idea: Adapt realizer for drinker's paradox

Define

$$\mathtt{Kif} := \begin{cases} \mathtt{0} & \mathtt{if}\ \mathtt{P_i}(\mathtt{0}) \vee \neg \mathtt{P_i}(\mathtt{f0}) \\ \mathtt{f0} & \mathtt{otherwise} \end{cases}$$

CLAIM: K witnesses $(*)$

$\boxed{0}$ First try $k = 0$. If $P_i(0) \vee \neg P_i(f0)$ stop, else

$\boxed{1}$ Try $k = f0$. Done.

Dialectica interpretation of classical analysis

$$\exists \chi \forall i, k \, . \, P_i(\chi i) \vee \neg P_i(k)$$

Dialectica interpretation of classical analysis

$$\neg\neg\exists\chi\forall i,k\;.\;P_i(\chi i)\vee\neg P_i(k)$$

Dialectica interpretation of classical analysis

$$(*) \quad \forall F, G \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \exists \chi .\, P_{F\chi}(\chi(F\chi)) \vee \neg P_{F\chi}(G\chi)$$

Dialectica interpretation of classical analysis

$$(*) \quad \forall F, G \colon (\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \exists \chi .\, P_{F\chi}(\chi(F\chi)) \vee \neg P_{F\chi}(G\chi)$$

Idea: build finite approximations to χ

Suppose u is a partial function with finite domain satisfying $\forall i \in \mathrm{dom}(u)\,.\, P_i(ui)$. Recursively define:

$$\chi[\mathtt{u}] := \begin{cases} \hat{\mathtt{u}} & \texttt{if } \mathtt{F}\hat{\mathtt{u}} \in \mathtt{dom}(\mathtt{u}) \texttt{ or } \mathtt{P}_{\mathtt{F}\hat{\mathtt{u}}}(\mathtt{0}) \vee \neg \mathtt{P}_{\mathtt{F}\hat{\mathtt{u}}}(\mathtt{G}\hat{\mathtt{u}}) \\ \chi[\mathtt{u} \oplus (\mathtt{F}\hat{\mathtt{u}}, \mathtt{G}\hat{\mathtt{u}})] & \texttt{otherwise}. \end{cases}$$

CLAIM: $\chi[\emptyset]$ witnesses $(*)$.

$\boxed{0}$ First try

$$\chi_0 := 0, 0, 0, \ldots.$$

Let $n_0, x_0 := F\chi_0, G\chi_0$. If $n_0 \in \emptyset$ or $P_{n_0}(0) \vee \neg P_{n_0}(x_0)$ stop, else...

$\boxed{1}$ Try

$$\chi_1 := 0, 0, \ldots, 0, \underbrace{x_0}_{n_0}, 0, \ldots.$$

Let $n_1, x_1 := F\chi_1, G\chi_1$. If $n_1 \in \{n_0\}$ or $P_{n_1}(0) \vee \neg P_{n_1}(x_1)$ stop, else...

$\boxed{2}$ Try

$$\chi_2 := 0, 0, \ldots, 0, \underbrace{x_0}_{n_0}, 0, \ldots, 0, \underbrace{x_1}_{n_1}, 0, \ldots.$$

Let $n_2, x_2 := F\chi_2, G\chi_2$. If $n_2 \in \{n_0, n_1\}$ or $P_{n_2}(0) \vee \neg P_{n_2}(x_2)$ stop, else...

Algorithm terminates for some χ_n by e.g. continuity of F, G.

Have transformed constructive interpretation of Σ^0_1-LEM:

$$\mathtt{Kif} := \begin{cases} \mathtt{0} & \mathtt{if}\ \ \mathtt{P_i}(\mathtt{0}) \vee \neg \mathtt{P_i}(\mathtt{f0}) \\ \mathtt{f0} & \mathtt{otherwise} \end{cases}$$

Pointwise learning procedure via single mind-changes.

to a constructive interpretation of Σ^0_1-arithmetical comprehension:

$$\chi\mathtt{FG}[\mathtt{u}] := \begin{cases} \hat{\mathtt{u}} & \mathtt{if}\ \ \mathtt{F}\hat{\mathtt{u}} \in \mathtt{dom}(\mathtt{u})\ \ \mathtt{or}\ \ \mathtt{P}_{\mathtt{F}\hat{\mathtt{u}}}(\mathtt{0}) \vee \neg \mathtt{P}_{\mathtt{F}\hat{\mathtt{u}}}(\mathtt{G}\hat{\mathtt{u}}) \\ \chi[\mathtt{u} \oplus (\mathtt{F}\hat{\mathtt{u}}, \mathtt{G}\hat{\mathtt{u}})] & \mathtt{otherwise}. \end{cases}$$

Global learning procedure via unbounded number of mind-changes.

Is this an instance of a more general transformation $K \to \chi$ which realizes the axiom of choice? Claim:

$$\chi FG[u] = \mathsf{SymBR}(F, G, K, u)$$

where for arbitrary F, G and K

$$\mathsf{SymBR}(F, G, K, u) = \begin{cases} \hat{u} & \text{if } F\hat{u} \in \mathrm{dom}(u) \\ \mathsf{SymBR}(F, G, K, u \oplus (F\hat{u}, c_u)) & \text{otherwise} \end{cases}$$

for $c_u := K(F\hat{u}, \lambda x\,.\,G(\mathsf{SymBR}(F, G, K, u \oplus (F\hat{u}, x))))$.

Background

This is a symmetric variant of Spector's well-known bar recursor (1962), investigated by P. Oliva and T.P. (2015).

$$\text{bar recursion} \quad : \quad \{\text{finite LPs } K_i\} \to \text{global LP } \bigotimes K_i$$

Intuition still needs to be made precise!