Modes of Bar Recursion

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(based on joint work with Martín Escardó and Paulo Oliva)
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3 July 2012

Outline

1. Introduction
   - Bar recursion
   - Overview of talk
2. Modes of bar recursion
   - PS / Finite bar recursion
   - EPS / Spector's bar recursion
   - IPS / Modified bar recursion
   - UR / Berardi-Bezem-Coquand functional
3. Interdefinability results
   - Relationship between EPS and IPS
   - Relationship between IPS and UR

Bar recursion

Primitive recursion: Recursion over the natural numbers. For $n \in \mathbb{N}$:

$$\mathrm{R}(n) := \begin{cases} y & \text{if } n = 0 \\ z_{n-1}(\mathrm{R}(n-1)) & \text{otherwise} \end{cases}$$

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Bar recursion: Recursion over well-founded trees. For $s \in T$:

$$B(s^{X^*}) := \begin{cases} Y_s & \text{if } s \text{ is a leaf} \\ Z_s(\lambda x \,.\, B(s * x)) & \text{otherwise} \end{cases}$$

Bar recursion is the wrong way round: $B(s)$ looks at the values of $B(s * x)$ for *extensions* of s!

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  Arithmetic (induction) $\mapsto$ System T (primitive recursion)

  Spector 1962 Dialectica interpretation of analysis

  Arithmetic + Countable choice $\mapsto$ System T + Bar recursion

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- **Escardó/Oliva 2010-** Product of selection functions. Interdefinability results. Links with game theory made precise.

Theme of talk

Computational aspects of bar recursion

1. Key computational features of different modes of bar recursion.
2. The relative strength of these modes of bar recursion.

To a lesser extent: The semantics of bar recursion (links with language of sequential games).

Why is this important?

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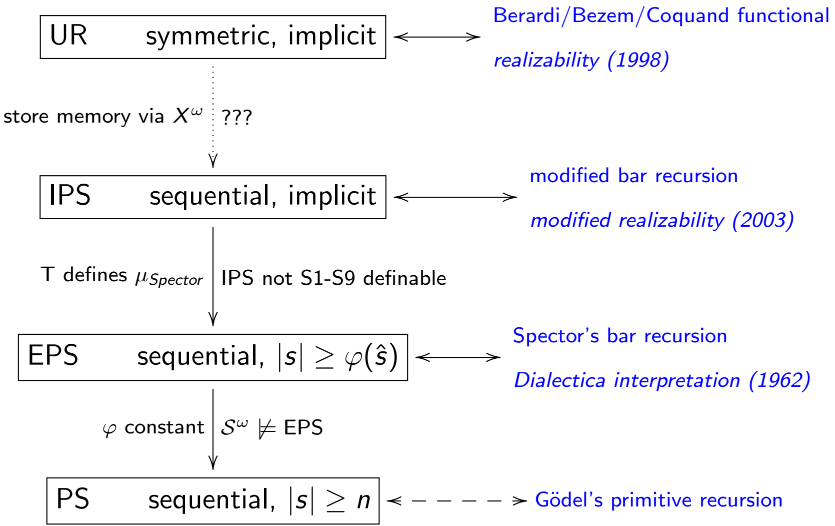
Why is this important?

- Open questions about an important class of non-primitive recursive functionals.
- Better understand computational content of classical proofs.
- Interesting mathematical problem.

Modes of bar recursion



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The finite product of selection functions

Idea: Sequential game with n rounds. Moves of type X, outcome of type R.

- $q\colon X^n \to R$ determines the *outcome* of a play of type X^n (by instead considering $q\colon X^\omega \to R$ type independent of n).
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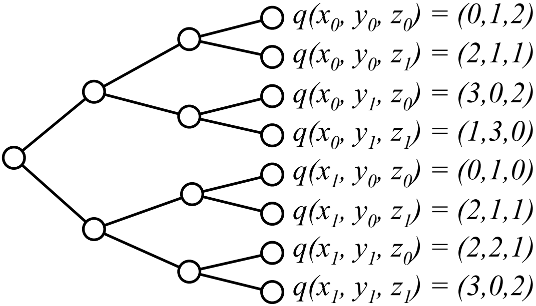
$$\mathrm{PS}^{\varepsilon,q,n}(s^{X^*}) \overset{X^*}{:=} \begin{cases} \langle\rangle & \text{if } |s| \geq n \\ a_s * \mathrm{PS}(s * a_s) & \text{otherwise} \end{cases}$$

where $a_s := \varepsilon_s(\lambda x\ .\ q(s * x * \mathrm{PS}(s * x)))$.

- For $|s| < n$, $\mathrm{PS}(s)$ is the optimal continuation of (of length $n - |s|$) of the play s.
- $\mathrm{PS}(\langle\rangle)$ is an optimal strategy for the game.

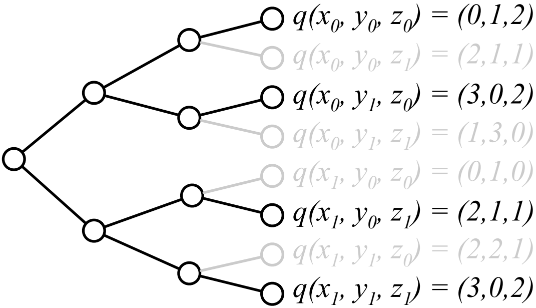
Example 1

$$X = [2] \quad ; \quad R = \mathbb{N} \quad ; \quad n = 3 \quad ; \quad q \colon [2]^3 \to \mathbb{N}^3 \quad ;$$
$$\varepsilon_i(p^{[2]\to\mathbb{N}}) = x \text{ maximising } p(x)_i$$



Example 1

$$\mathrm{PS}(x_1, y_0) = \langle \varepsilon_2(z_0 \mapsto 0, z_1 \mapsto 1)\rangle * \mathrm{PS}(x_1, y_0, z_i) = \langle z_1 \rangle * \langle\rangle = \langle z_1 \rangle$$



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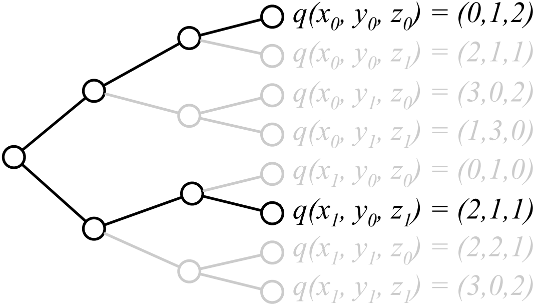

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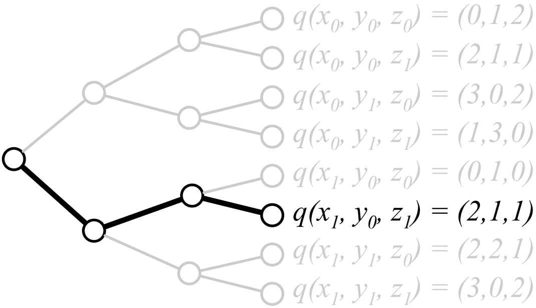

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- $\varepsilon_{2i}(p^{X \to R}) / \varepsilon_{2i+1}(p)$ selects x maximising/minimising $p(x)$.
- $\mathrm{PS}^{\varepsilon,q,n}(\langle\rangle)$ returns an 'optimal' play, resulting in a draw.

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Remark. PS is equivalent over a weak base theory to Gödel's primitive recursion in all finite types.

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Semantics. Operation that computes optimal strategies in finite sequential games.

Modes of bar recursion

PS	sequential, $\lvert s \rvert \geq n$

$\lessdot - \gtrdot$ Gödel's primitive recursion

Explicitly iterated product of selection functions (EPS)

Idea: Sequential game with unbounded number of rounds.

- $q \colon X^\omega \to R$ determines outcome of each infinite play X^ω.
- $\varepsilon_s \colon (X \to R) \to X$ dictates a *strategy* for $|s|$th round given any partial play s^{X^*}.
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Stopping condition now depends on s!

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$$\varphi(\widehat{[\lambda i.1](n)}) = \varphi(\underbrace{1, \ldots, 1}_{n \text{ times}}, 0, 0, \ldots) = n + 1 > n.$$

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By CONT, $\varphi(\widehat{[\alpha](n)}) = \varphi\alpha$ for all $n \geq N$, so for $n = \max\{N, \varphi\alpha\}$ we have $n \geq \varphi\alpha = \varphi(\widehat{[\alpha](n)})$.

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If $\mathrm{EPS}(s*x)$ well-defined for all extensions $s*x$ of s, then by definition so is $\mathrm{EPS}(s)$.

By the principle of *bar induction* $\mathrm{EPS}(\langle\rangle)$ is well-defined.

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Modes of bar recursion

EPS sequential, $|s| \geq \varphi(\hat{s})$ $\longleftrightarrow$ Spector's bar recursion

Dialectica interpretation (1962)

φ constant $\downarrow$ $\mathcal{S}^\omega \not\models$ EPS

PS sequential, $|s| \geq n$ $\longleftarrow - - - \rightarrow$ Gödel's primitive recursion

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No longer a stopping condition!

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Therefore $\mathsf{IPS}([\alpha](N))_0 = \varepsilon_{[\alpha](N)}(\lambda x \,.\, q(\alpha))$ and by induction

$$\mathsf{IPS}([\alpha](N)) = \lambda k \,.\, \varepsilon_{[\alpha](N) * t_k}(\lambda x \,.\, q\alpha).$$

where $t_k = [\mathsf{IPS}([\alpha](N))](k)$, so $\mathsf{IPS}([\alpha](N))$ well-defined.

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Therefore $\mathrm{IPS}([\alpha](N))_0 = \varepsilon_{[\alpha](N)}(\lambda x \,.\, q(\alpha))$ and by induction

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By bar induction $\mathrm{IPS}(\langle\rangle)$ is well-defined.

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Semantics. Operation that computes optimal strategies in unbounded sequential games, only finite part of a play considered by continuity of q.

Modes of bar recursion

IPS sequential, implicit $\longleftrightarrow$ modified bar recursion

modified realizability (2003)

EPS sequential, $|s| \geq \varphi(\hat{s})$ $\longleftrightarrow$ Spector's bar recursion

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Update recursion (UR)

Idea: Compute a sequence, but not sequentially...

$$\mathrm{IPS}'(s) := s * \lambda k \,.\, \varepsilon_{s*t_k}(\lambda x \,.\, q(\mathrm{IPS}'(s * t_k * x)))$$

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Recursion is no longer sequential!

Why is UR well defined?

UR(u) requires us to know value of UR(u_k^x) for all *updates* of u (not just extension), so not clear how we can use bar induction to show that UR is total...

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An *open predicate* on sequences X^ω is one of the form $A(\alpha) = \exists N\, B([\alpha](N))$.

Definition. Update induction is given by the schema

$$\forall u(\forall n \notin \mathrm{dom}(u), x\ A(u_n^x) \to A(u)) \to \forall u\ A(u).$$

Update induction follows from dependent choice.

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By update induction $q(\mathrm{UR}(u))$ total for all u, and therefore $\mathrm{UR}(u)$ is total for all u.

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Semantics. Can be viewed as computing an optimal strategy in games where players 'ignore' the others. *Game theoretic semantics not properly formalised for* UR*!*

Modes of bar recursion

UR — symmetric, implicit	⟷	Berardi/Bezem/Coquand functional *realizability (1998)*
IPS — sequential, implicit	⟷	modified bar recursion *modified realizability (2003)*
EPS — sequential, $\lvert s\rvert \geq \varphi(\hat{s})$	⟷	Spector's bar recursion *Dialectica interpretation (1962)*
↓ φ constant, $\mathcal{S}^\omega \not\models$ EPS		
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Outline

1. Introduction
   - Bar recursion
   - Overview of talk
2. Modes of bar recursion
   - PS / Finite bar recursion
   - EPS / Spector's bar recursion
   - IPS / Modified bar recursion
   - UR / Berardi-Bezem-Coquand functional
3. Interdefinability results
   - Relationship between EPS and IPS
   - Relationship between IPS and UR

T-definablility

Definition A functional Ψ is T-definable from a functional Φ over a theory $\mathcal{S}$ (we write $\mathcal{S} \vdash \Phi \geq_{\mathrm{T}} \Psi$) if there exists a term t in system T such that $t(\Phi)$ satisfies the defining equation of Ψ provably in $\mathcal{S}$.

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In general our theory $\mathcal{S}$ will be something like $\mathrm{HA}^\omega + \mathrm{CONT} + \mathrm{BI}$.

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Key observation: The so-called Spector's search functional

$$\mu_{Sp}(\varphi^{X^\omega \to \mathbb{N}})(\alpha^{X^\omega}) := \text{least } n\ (n \geq \varphi(\widehat{[\alpha](n)}))$$

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Can encode stopping condition $|s| \geq \varphi(\hat{s})$ into $\tilde{\varepsilon}$, $\tilde{q}$ such that $\text{IPS}^{\tilde{\varepsilon},\tilde{q}}$ T-defines EPS.

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FAN is S1-S9 + IPS computable in $\mathcal{C}^\omega$

$\Rightarrow$ IPS is not S1-S9 computable in $\mathcal{C}^\omega$

$\Rightarrow$ IPS is not T-definable from EPS in any theory that has a model in $\mathcal{C}^\omega$.

Modes of bar recursion

UR symmetric, implicit ⟷ Berardi/Bezem/Coquand functional *realizability (1998)*

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T defines $\mu_{Spector}$ | IPS not S1-S9 definable

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Key idea: Use UR to compute a sequence of *sequences*, i.e. moves of type X^ω. Entries may be computed independently, but using sequence types allows us to store recursive calls.

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How do we simulate a sequential algorithm like IPS with a non-sequential algorithm like UR?

Key idea: Use UR to compute a sequence of *sequences*, i.e. moves of type X^ω. Entries may be computed independently, but using sequence types allows us to store recursive calls.

A fairly complex construction with a long and tedious verification. Won't go into any more detail!

Does IPS T-define UR?

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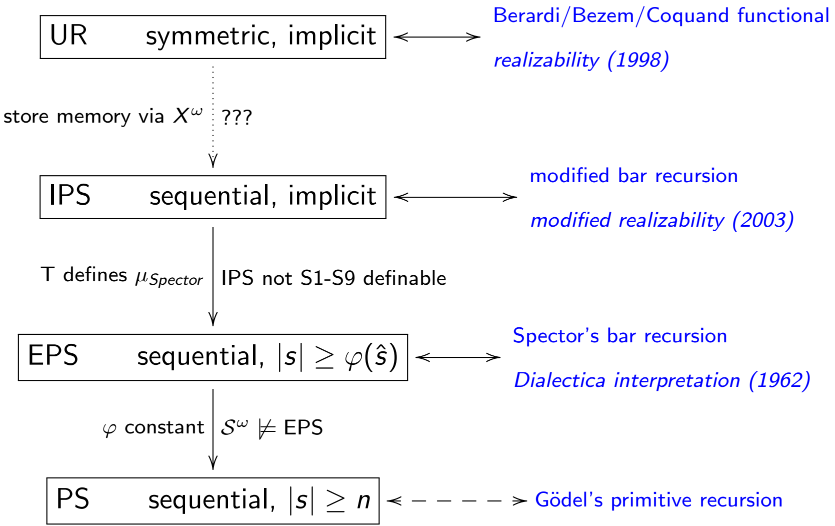
Does IPS T-define UR?

Can we simulate a non-sequential algorithm like UI with a sequential algorithm like IPS?

Unknown!

Modes of bar recursion



Further questions

The key difference between UR and IPS...

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Can we generalise this to associate a form of bar recursion to an arbitrary tree $\prec$?

How is this family of bar recursion functionals related?

New realisers for program extraction?

Direction for future work

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- Develop some new results and machinery in theory of higher-type computability.

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