System T and the Product of Selection Functions

Thomas Powell
(joint work with Paulo Oliva and Martín Escardó)

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7 September, 2011

Outline

1. Fragments of arithmetic
2. The product of selection functions
3. Fragments of system **T**
4. Selection functions in analysis

Induction and recursion

Any strong theory of arithmetic proves some form of induction

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In this talk, a strong theory of (higher-type) functionals allows construction of functions using primitive recursion

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Primitive recursion is the computational analogue of induction.

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- The strong fragment $\mathbf{T}_n$ consists of $\mathbf{T}_{\mathrm{b}}$ along with primitive recursors for types of degree $\leq n$.
- System **T** consists of $\mathbf{T}_{\mathrm{b}}$ along with primitive recursors of all finite types.

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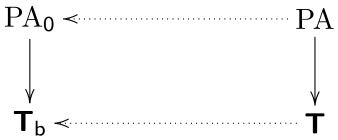

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Parsons (1972): $\mathrm{I}\Sigma_{n+1}$ has a functional interpretation in $\mathbf{T}_n$.

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Suppose that for some colouring $f : \mathbb{N} \rightarrow [m]$, each colour is used only finitely many times i.e. $\forall i \leq m \exists x \forall y \geq x (f(y) \neq i)$.

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but this fails for $y := \max \alpha_i$, $i := f(\max \alpha_i)$.

Therefore some colour is used infinitely often.

An alternative arithmetic hierarchy

Fragments of Peano arithmetic based on choice

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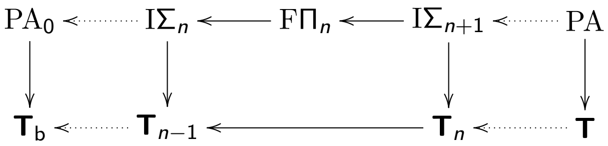

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Countable choice	$\rightsquigarrow$	Spector's bar recursion
	$\rightsquigarrow$	Unbounded product of selection functions *Optimal strategies in unbounded games*
Finite choice	$\rightsquigarrow$	Finite product of selection functions *Optimal strategies in finite games*

The product of selection functions

Given $\varepsilon_i \colon (X \to Y) \to X$, $q \colon X^{\mathbb{N}} \to Y$ define

$$\mathrm{P}_i^{X,Y}(\varepsilon)(m)(q) \stackrel{X^{\mathbb{N}}}{=} \begin{cases} \mathbf{0}^{X^{\mathbb{N}}} & \text{if } i > m \\ a * \mathrm{P}_{i+1}^{X,Y}(\varepsilon)(m)(q_a) & \text{otherwise} \end{cases}$$

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- A sequential game with $m+1$ rounds;
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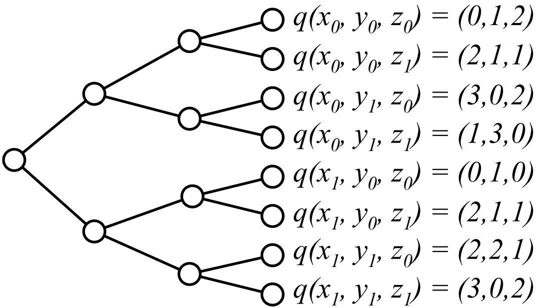
- ε_i determines the strategy at round i;
- p_i maps potential plays x to optimal outcome.

Illustration

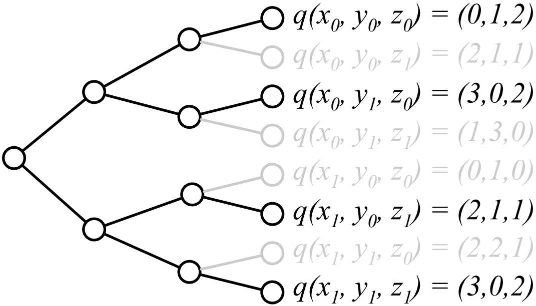
$$\varepsilon_i p = \max\ (\pi_i \circ p)x \text{ for } i = 0, 2$$

$$\varepsilon_i p = \min\ (\pi_i \circ p)x \text{ for } i = 1$$



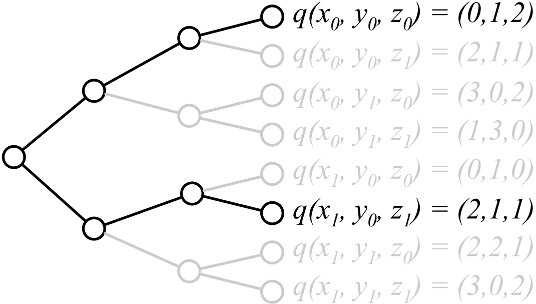
Illustration

$$\mathsf{P}_2(\varepsilon)(2)(q_{x_1,y_0}) = \langle z_1 \rangle$$



Illustration

$$\mathsf{P}_1(\varepsilon)(2)(q_{x_1}) = \langle y_0, z_1 \rangle$$



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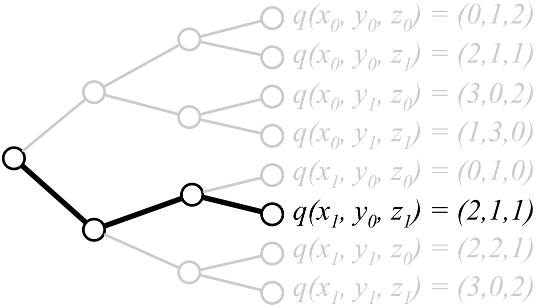

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There exists a selection function $\varepsilon \colon (X \to Y) \to X$ that for any counterexample function $p \colon X \to Y$ selects a point at which it fails i.e. $A(\varepsilon p, p(\varepsilon p))$ holds.

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Premise: there exists a collection (ε_i) of strategies refuting **pointwise** counterexample functions p_i for A_i.

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Conclusion: there exists a co-operative strategy α_q refuting a **global** counterexample function q for $\forall i \leq m A_i$.

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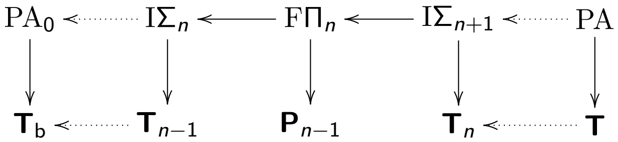

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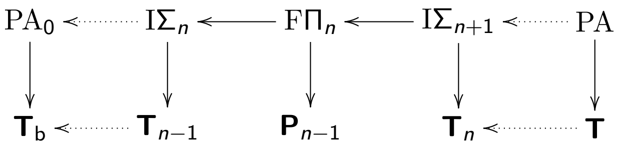

What is the relationship between Gödel's primitive recursors and the product of selection functions?

Outline

A primitive recursive definition of the product

$$\mathsf{P}_i^{X,Y}(\varepsilon)(m)(q) \overset{X^{\mathbb{N}}}{=} \begin{cases} \mathbf{0}^{X^{\mathbb{N}}} & \text{if } i > m \\ a * \mathsf{P}_{i+1}^{X,Y}(\varepsilon)(m)(q_a) & \text{otherwise} \end{cases}$$

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We can define $\mathrm{P}_0^{X,Y}(\varepsilon)(m)(q)$ using primitive recursion of type $X^* \to X^{\mathbb{N}}$:

$$y := \lambda s.\mathbf{0}^{X^{\mathbb{N}}}$$

$$z(i, F^{X^* \to X^{\mathbb{N}}}) := \lambda s.a_s * F(s * a_s).$$

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Claim: $\mathrm{P}_0^{X,Y}(\varepsilon)(m)(q) = \mathrm{R}_{m+1}(y, z)(\langle\rangle)$.

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$$y := \lambda s.\mathbf{0}^{X^{\mathbb{N}}}$$

$$z(i, F^{X^* \to X^{\mathbb{N}}}) := \lambda s.a_s * F(s * a_s).$$

Claim: $\mathrm{P}_0^{X,Y}(\varepsilon)(m)(q) = \mathrm{R}_{m+1}(y, z)(\langle\rangle)$.

Theorem

$\mathbf{T}_{n+1} \Rightarrow \mathbf{P}_n$ over $\mathbf{T}_{\mathrm{b}}$.

Computations on a register machine

Products of type X of the form $\mathsf{P}^{X,X^{\mathbb{N}}}(\varepsilon)(m)(id)$ are canonical.

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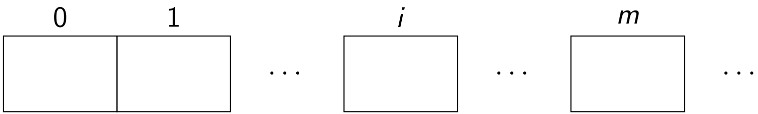

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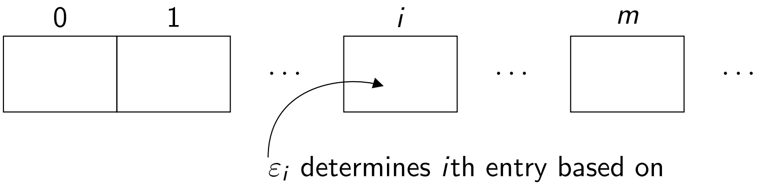

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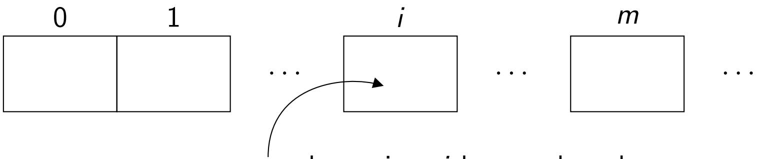

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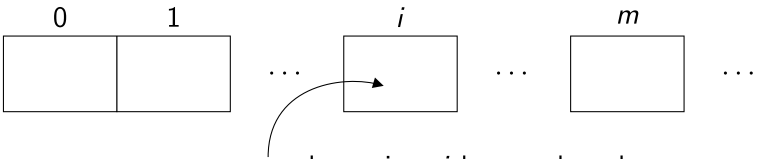

ε_i determines ith entry based on

1. previous entries
2. effect choices have on *subsequent* entries

Simulating primitive recursion

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\begin{aligned}
R_0(y, z) &= y \\
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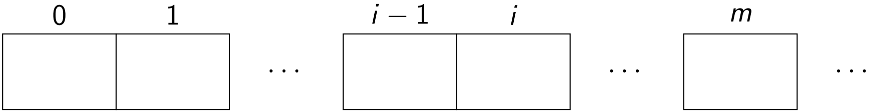

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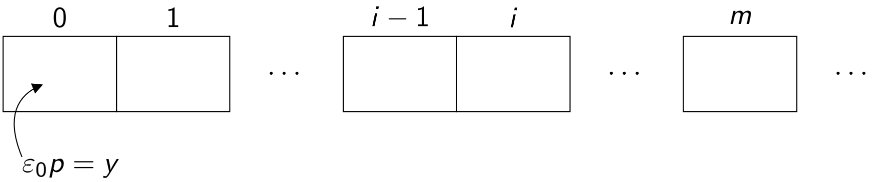

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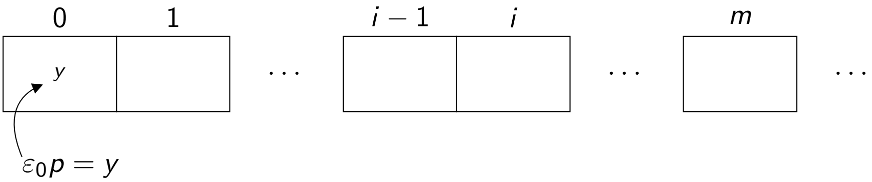

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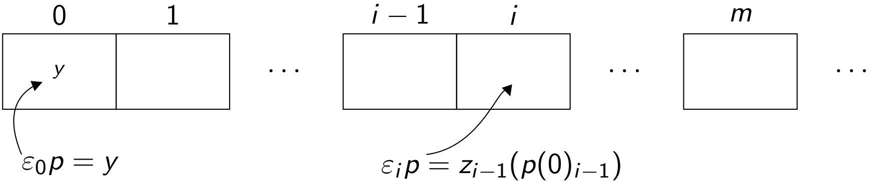

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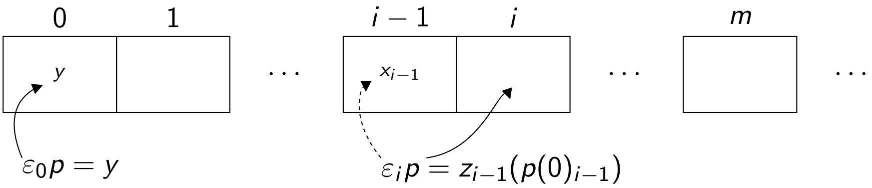

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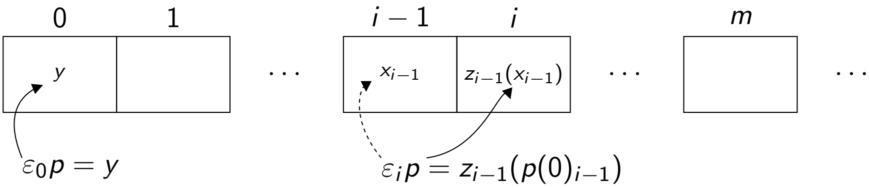

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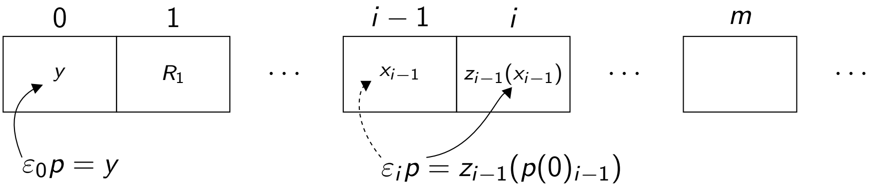

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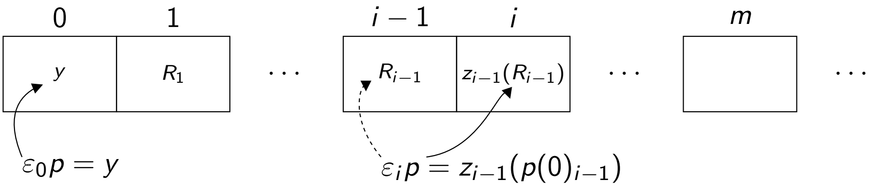

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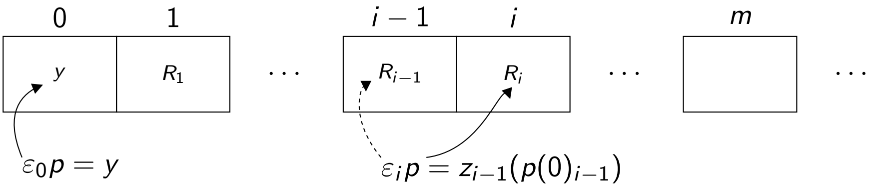

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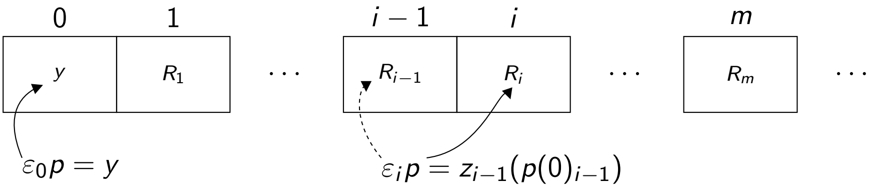

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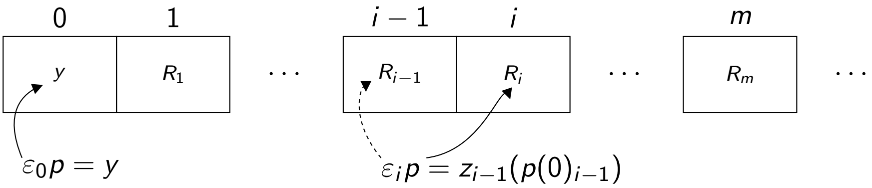

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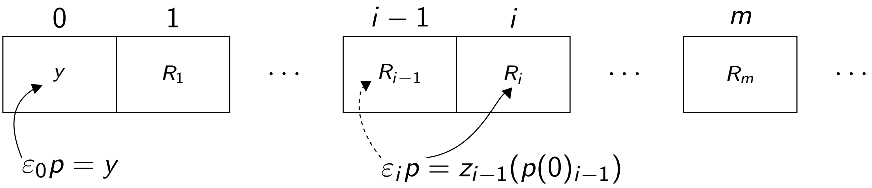

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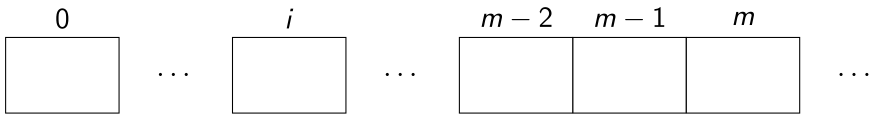

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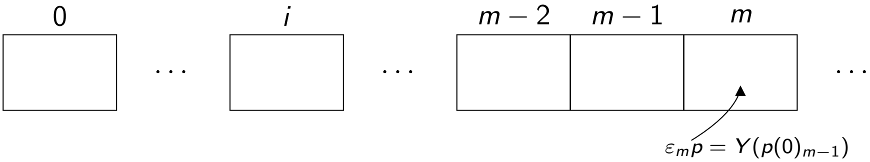

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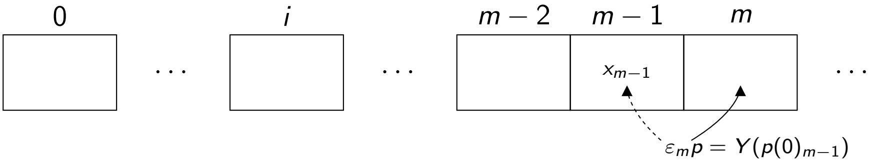

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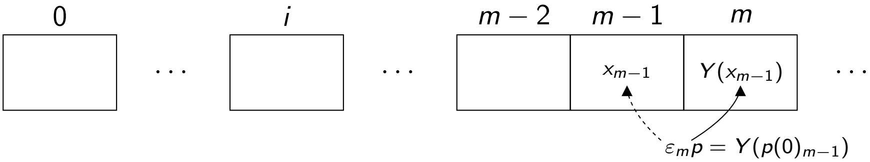

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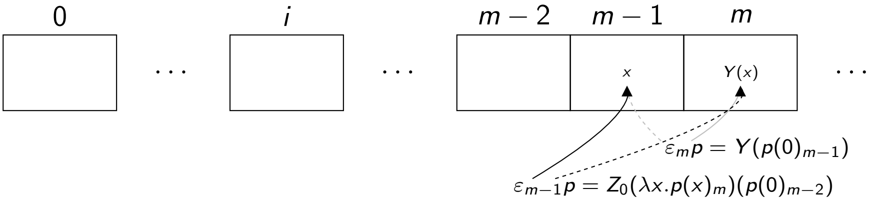

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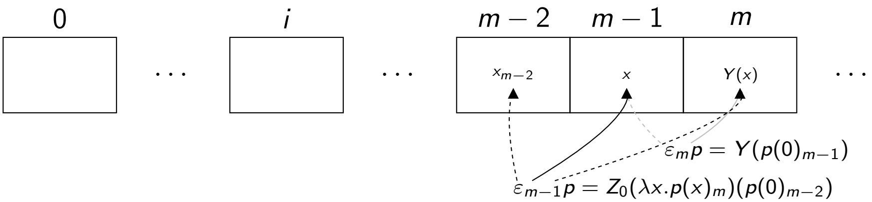

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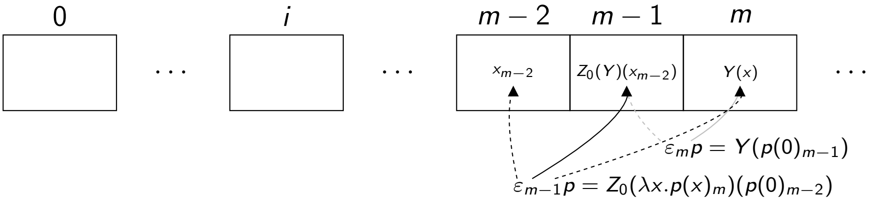

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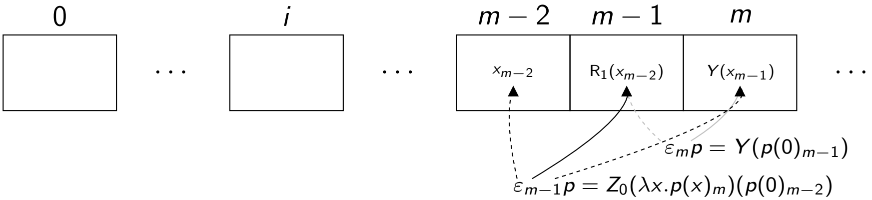

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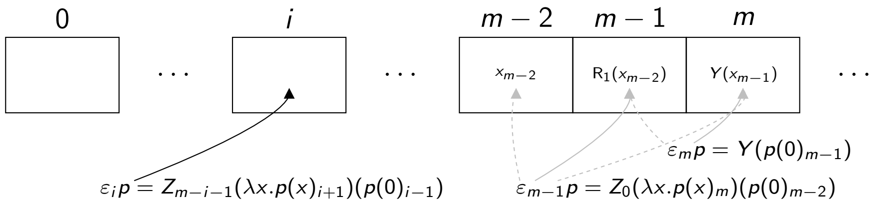

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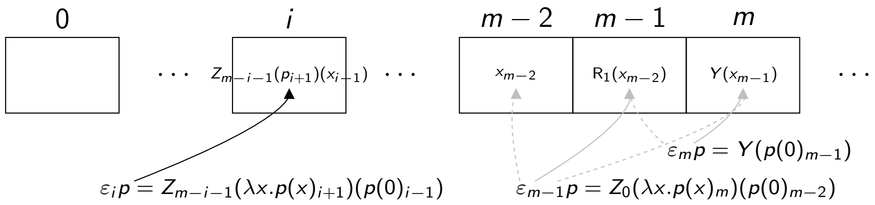

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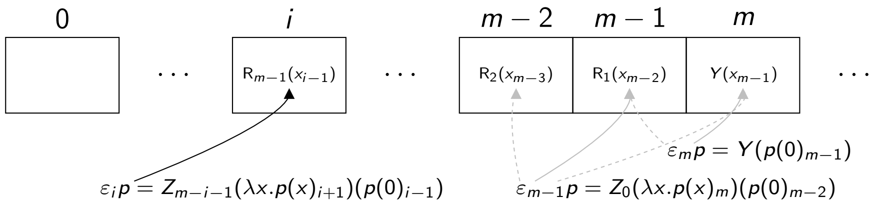

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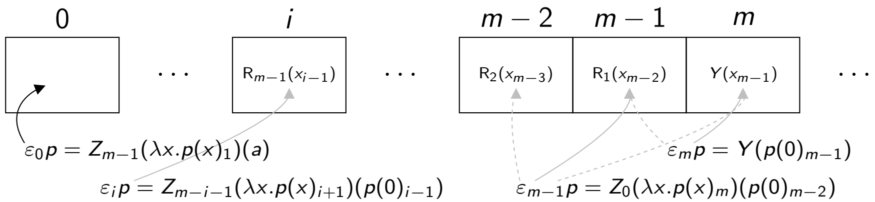

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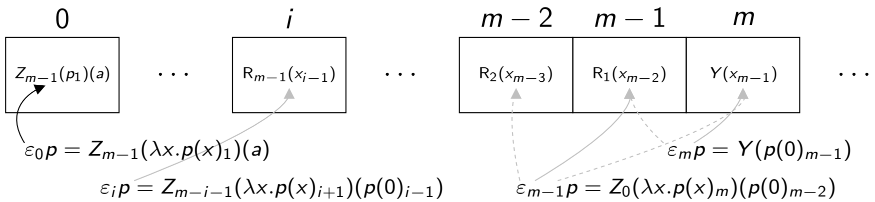

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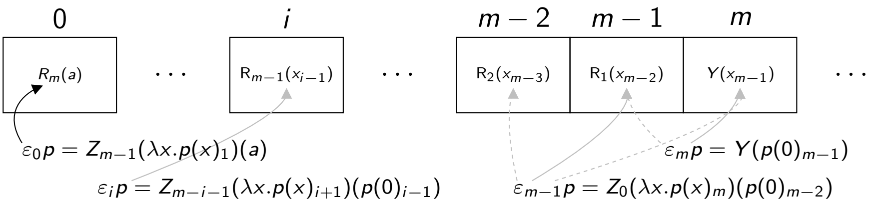

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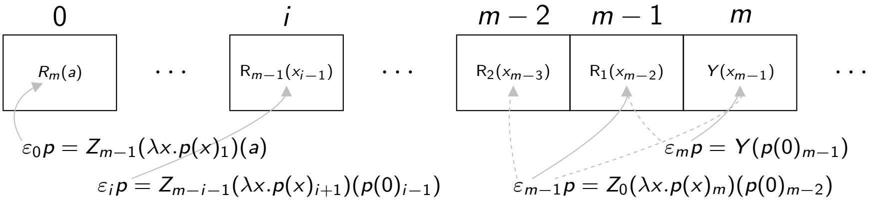

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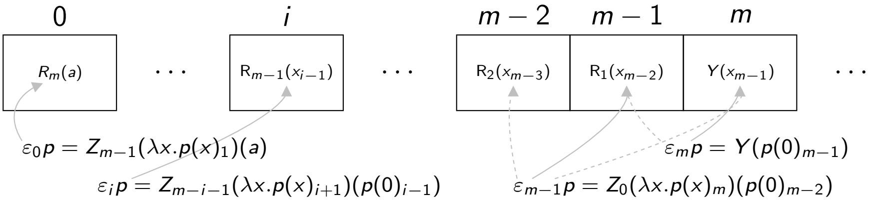

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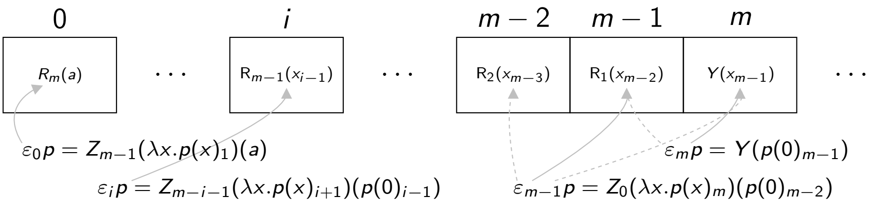

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Does $\mathrm{F}\Pi_n$ have a functional interpretation is a fragment weaker than $\mathbf{P}_{n-1}$?

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- Resulting fragments of **T** correspond to fragment of arithmetic based on finite choice, as opposed to induction.
- An unbounded version of the product is equivalent to Spector's bar recursion: uniform transition from arithmetic to analysis.

$$\begin{array}{ccc} \text{Arithmetic} & \rightsquigarrow & \text{Finite games} \\ \text{Analysis} & \rightsquigarrow & \text{Unbounded games} \end{array}$$

Outline

A computational analogue of finite choice

Countable choice $\rightsquigarrow$ Spector's bar recursion

Coquand et al. (1998), Oliva and Escardo (2009):
Computational content of choice has game theoretic character.

Countable choice $\rightsquigarrow$ Unbounded product of selection functions
Optimal strategies in unbounded games

The unbounded product of selection functions

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$$\mathrm{EPS}_i^{X,Y}(\varepsilon)(\omega)(q) \stackrel{X^{\mathbb{N}}}{=} \begin{cases} \mathbf{0}^{X^{\mathbb{N}}} & \text{if } \omega\alpha < i \\ a * \mathrm{EPS}_{i+1}^{X,Y}(\varepsilon)(\omega)(q_a) & \text{otherwise} \end{cases}$$

where $a := \varepsilon_i(\underbrace{\lambda x . q_x(\mathrm{EPS}_{i+1}^{X,Y}(\varepsilon)(\omega)(q_x))}_{p_i})$.

A game-theoretic interpretation of analysis

A large portion of analysis can be formalised in Peano arithmetic plus countable choice.

Theorem

$\mathrm{PA} + \mathsf{AC}^0$ has a functional interpretation in $\mathbf{T} + \mathsf{EPS}$.

Theorems in mathematical analysis have an intuitive computational interpretation in terms of optimal strategies in sequential games

Why are we interested in the qualitative behaviour of functional interpretations?

The correspondence principle

T. Tao: Correspondence between ‘hard’ and ‘soft’ analysis.

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Theorem (Bounded convergence principle)

Given $\varepsilon > 0$, $0 \leq x_0 \leq x_1 \leq \ldots \leq 1$, *there exists* n *such that* $|x_{n+m} - x_n| \leq \varepsilon$ *for all* m.

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Fix $F \colon \mathbb{N} \to \mathbb{N}$. *Given* $\varepsilon > 0$, $0 \leq x_0 \leq \ldots x_M \leq 1$, *if* M *sufficiently large exists* N *s.t.* $|x_{N+F(N)} - x_N| \leq \varepsilon$.

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Permanent stability vs. arbitrary high quality regions of metastability

Functional interpretations as 'finitizations'

Kohlenbach: This is what the 'monotone' functional interpretation does.

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monotone functional interpretation	$\Leftrightarrow$	correspondence principle
higher-type realiser of interpreted formula	$\Leftrightarrow$	constructive proof of finitized theorem

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- Operational behaviour of extracted algorithms easier to understand in terms of a constructive mathematical proof of an interpreted/finitised theorem.

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Proof.

$$\varepsilon_n p := \begin{cases} \text{don't add } n & \text{if } \forall i \leq p0\ \neg\varphi_0(n, p0) \\ \text{add } n & \text{otherwise} \end{cases}$$

In practise construct Y recursively: $\emptyset \mapsto Y_1 \mapsto Y_2 \mapsto \ldots \mapsto Y$ each iteration adding a discovered element of X until Y_i large enough. □

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- Want to bridge the gap between formal program extraction and practical mathematics.