A finitization of Littlewood's Tauberian theorem and an application in Tauberian remainder theory

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Abstract

In this paper we study Littlewood's Tauberian theorem from a proof theoretic perspective. We first use the Dialectica interpretation to produce an equivalent, finitary formulation of the theorem, and then carry out an analysis of Wielandt's proof to extract concrete witnessing terms. We argue that our finitization can be viewed as a generalized Tauberian remainder theorem, and we instantiate it to produce two concrete remainder theorems as a corollary, in terms of rates of convergence and rates metastability, respectively. We rederive the standard remainder estimate for Littlewood's theorem as a special case of the former.

1 Introduction

The extraction of computational content from proofs is a central theme in logic and theoretical computer science. Modern research on this topic encompasses both foundational results (such as the correspondence between formal logic and programming languages, the computational semantics of proofs, and complexity theory), along with applications (including formal verification, and the use of logical techniques to obtain numerical data from proofs in mathematics). This paper forms a new contribution to the application of proof theory in mathematics, using Gödel's Dialectica interpretation [6] to give a computational interpretation to Littlewood's classic Tauberian theorem [17], and then analysing a proof of the theorem to produce witnesses for its interpretation.

Ever since the pioneering work of Kreisel [15, 16] on the "unwinding" of proofs, it has been clear that traditional logical methods, and in particular proof interpretations such as the Dialectica, have a deep mathematical significance. This has been widely demonstrated in the last 30 years or so through the *proof mining* program [10], which makes use of proof theoretic techniques to not only obtain new quantitative information from nonconstructive proofs, but also yield qualitative generalisations of theorems, along with deep structural insights into certain mathematical phenomena. An example of the latter is the recent discovery [5] that the Dialectica interpretation is connected to a fundamental correspondence principle between 'soft' and 'hard' statements in analysis as described by T. Tao [25], and that the interpretation can be viewed as a method for ‘finitizing’ infinitary statements, particularly convergence properties. Thus using the Dialectica interpretation to explore computational aspects of mathematical proofs can result in both useful numerical results and interesting connections between logic and pure mathematics.

In this paper, we apply the Dialectica interpretation in this spirit to investigate the relationship between two forms of convergence, representing distinct methods for summing an infinite sequence of real numbers $\{a_n\}$:

$$(i)\quad \sum_{i=0}^{\infty} a_i \quad \text{and} \quad (ii)\quad \lim_{x\to 1^-}\sum_{i=0}^{\infty} a_i x^i$$

When the limit (ii) exists, we say that $\{a_n\}$ is *Abel summable*. While normal summability (i) implies Abel summability (ii), the converse is not true, although a partial converse can be proven by imposing a growth condition on $\{a_n\}$. In 1897 A. Tauber first showed that $a_n = o(1/n)$ suffices, but then in 1911 J. E. Littlewood established an optimal order condition such that convergence of (ii) implies (i), namely $a_n = \mathcal{O}(1/n)$ [17]. This celebrated theorem marked the beginning of the lifelong collaboration between G. H. Hardy and Littlewood, which began with the development of further theorems of Tauberian type, and ultimately launched an area of analysis now knows as Tauberian theory [14].

It turns out that the proof theoretic analysis of Tauberian theorems gives us interesting results. The Dialectica interpretation of these theorems can be formulated in a very natural way as an implication between the appropriate ‘metastable’ variants of the relevant convergence properties – and when the latter are put in Cauchy form, the result is an elegant finitization along the lines of Tao’s finite convergence principle [25]. However, obtaining the corresponding bounds involves a careful analysis of the original proofs. Crucially, this process of finitization and bound extraction has a relevance beyond mathematical logic: So-called remainder estimates, which relate the convergence speed of different methods of summability, have been widely studied in parallel with the development of Tauberian theorems, forming a fascinating quantitative subfield of the area (cf. [14, Chapter VII]). Our Dialectica interpretation of Littlewood’s theorem can be viewed as a generalised remainder theorem: Not only are we able to rederive the canonical remainder estimate as a special case, but our finitary theorem can be used to obtain meaningful numerical results that go beyond the scope of traditional Tauberian remainder theory, for example, cases where the summability methods do not even possess computable rates of convergence.

Above all, in this paper we build on our initial work [22] and present a case for Tauberian theory as a potentially fruitful area of application for proof theoretic techniques in general. Over the course of the last century, Tauberian theory has been vastly expanded beyond Littlewood’s theorem, and we conjecture that the proof-theoretic analysis of more complex Tauberian theorems could form an interesting contribution to both applied proof theory and Tauberian remainder theory. For example, we believe that several of the key ideas presented here could subsequently be lifted to a more general setting and used to finitize e.g.

integral analogues of Tauberian theorems using Karamata's method.

To summarise, then, there are three main reasons why we have chosen Littlewood's theorem as a candidate for a proof theoretic study:

1. The theorem is extremely simple to state, and from a logical perspective can be reformulated as a straightforward implication between Cauchy convergence properties (Section 3). Proofs of the theorem, on the other hand, are complex, and even the shortest method of proof discovered by Karamata [8] involves subtle ideas from approximation theory. As such, the Dialectica interpretation of Littlewood's theorem is elegant and intuitive, but extracting the corresponding witnessing terms poses a challenge (Section 5).

2. There is an existing interest in quantitative versions of Tauberian theorems in the form of remainder estimates, which relate the convergence speed of (ii) to that of (i). We are able to not only provide a generalisation of the known estimate in the case of Littlewood's theorem, but also show that the latter falls out in a natural way from our proof-theoretic analysis (Section 6).

3. Tauberian theory in general represents a new domain of application for proof theoretic methods, and is an area replete with simple convergence statements whose proofs make use of complex analytic techniques. As such, an analysis of Littlewood's theorem represents a step in a promising new direction with great potential for further study. Concrete suggestions for future research are given in the conclusion to this paper.

Though the central theme of this paper is the application of proof theoretic methods in mathematics, we also consider our results to be of broader relevance to the mathematical logic community: Through our use of the Dialectica interpretation we are able impose a natural game semantics onto Tauberian theorems, along the lines of [2, 3], with our witnessing terms corresponding to a winning strategy; The paper as a whole requires a careful analysis of the logical structure of convergence properties that could be of interest to researchers in constructive mathematics or formalization; Our remainder theorems in Section 6 involve concepts from computability theory such as Specker sequences, and are formulated in terms of higher-order functionals, so that Theorem 6.7 is essentially the specification of a type 3 functional program corresponding to a variant of Littlewood's theorem.

On a more general level, in recent years there has been a resurgence of interest in the Dialectica interpretation outside of applied proof theory, encompassing formalization [23], category theory (starting with [4]), and most recently new connections with classical realizability [19] and the differential lambda calculus [9]. Therefore we believe that a self-contained case study presenting a novel example of how the Dialectica interpretation manifests in a natural way within a beautiful area of pure mathematics will be of value in its own right.

For this reason, we have written the paper without assuming any prior knowledge of either Tauberian theory or proof interpretations. We provide a brief overview of Tauberian theory in the next section, and introduce the relevant aspects of the Dialectica interpretation as they are needed in later sections. Beyond this, we assume only a basic understanding of formal logic and a certain fluency in elementary analysis, specifically convergent series and integrals.

Related work

Tauberian theory has grown into a large area of research, but applications of proof theory in this area are currently limited to the author's previous work [22], on which this paper builds significantly (though in the other direction, Tauberian theorems have been applied in proof theory by Weiermann to derive ordinal bounds cf. [28] and most recently [29]). More generally, the results of this paper represent one of very few applications of proof interpretations that involve analytic methods in number theory (as broadly defined), beyond very early case studies such as [16] and [18].

2 Tauberian theorems

Tauberian theory is an extensive area of research which, taken in a very general sense, is concerned with finding conditions under which summability methods converge. For a comprehensive overview of the field, including an account of its historical development and a survey of modern research in the area, the reader is encourage to consult the standard textbook [14], though everything that we require will be presented below.

The present paper involves just two simple summability methods for sequences of real numbers, namely the basic infinite sum of their elements, together with the power series they generate. To be more specific, let $\{a_n\}$ be a sequence of real numbers. For the remainder of this paper we will define $\{s_n\}$ and $F : [0, 1) \to \mathbb{R}$ in terms of $\{a_n\}$ as follows:

$$s_n := \sum_{i=0}^{n} a_i \quad \text{and} \quad F(x) := \sum_{i=0}^{\infty} a_i x^i$$

Assuming that $\{|a_n|\}$ is bounded above by some $a > 0$, it is clear that F is well-defined on $x \in [0, 1)$, since then

$$\sum_{i=0}^{\infty} |a_i x^i| \leq a \sum_{i=0}^{\infty} x^i = \frac{a}{1-x}$$

However, the question of whether or not $F(x)$ converges to some finite limit as $x \to 1^-$ is closely related to the convergence of $\{s_n\}$. In one direction we have a standard result:

Theorem 2.1 (Abel's theorem)**.** *If* $\lim_{n\to\infty} s_n = s$ *and thus* $F(x)$ *is well-defined on* $[0, 1)$*, then* $\lim_{x\to 1^-} F(x) = s$.

However, the converse of this theorem is not true: Setting $a_n = (-1)^n$ we have

$$F(x) = \sum_{i=0}^{\infty} (-1)^i x^i = \frac{1}{1+x} \to \frac{1}{2}$$

as $x \to 1^-$, but $\sum_{i=0}^{\infty} (-1)^i$ does not exist. Tauber's theorem, from which Tauberian theory derives its name, establishes a simple growth condition under which a converse to Abel's theorem *does* hold:

Theorem 2.2 (A. Tauber 1897 [27]). *If* $\lim_{x \to 1^-} F(x) = s$ *and* $a_n = o(1/n)$ *then* $\lim_{n \to \infty} s_n = s$.

Both Abel's and Tauber's theorems can be easily proven with little more than elementary facts about convergent series (cf. [14, Chapter 1] and the corresponding proof theoretic analysis in [22]). However, in 1911, Littlewood established a 'big-$\mathcal{O}$' strengthening of Tauber's theorem, a much deeper result which in some sense marked the beginning of Tauberian theory in earnest:

Theorem 2.3 (J. E. Littlewood 1911 [17]). *If* $\lim_{x \to 1^-} F(x) = s$ *and* $a_n = \mathcal{O}(1/n)$ *then* $\lim_{n \to \infty} s_n = s$.

Littlewood showed that his growth condition $a_n = \mathcal{O}(1/n)$ is optimal in the sense that for any sequence $\{b_n\}$ with $\lim_{n \to \infty} b_n = \infty$ there exists a sequence $\{a_n\}$ with $n|a_n| \leq b_n$ such that $\lim_{x \to 1^-} F(x)$ exists but $\sum_{i=0}^{\infty} a_i$ does not, although together with Hardy [7] his Tauberian theorem was further strengthened, in particular showing that a one-sided condition $na_n \geq -C$ is sufficient.

A key feature that distinguishes both Littlewood's and subsequent Tauberian theorems from Theorem 2.2 is the relative difficulty of proving them. Littlewood's original proof was complex and involved repeated differentiation, and while much simpler proofs were subsequently found (notably by Karamata [8] and then Wielandt [30]), all of these rely on analytic methods, specifically techniques for approximating continuous functions with polynomials.

Another good indication that Littlewood-style Tauberian theorems go fundamentally beyond elementary analysis is that optimal remainder estimates that relate the convergence speed of $F(x) \to s$ to that of $s_n \to s$ are established using numerical results from approximation theory, including bounds on the degree and coefficients of polynomials that approximate piecewise continuous functions. The canonical remainder estimate for Littlewood's theorem is as follows:

Theorem 2.4 (cf. Korevaar [14] page 346). *Suppose that* $a_n = \mathcal{O}(1/n)$ *and there is some* $b > 0$ *such that*

$$F(x) = s + \mathcal{O}((1-x)^b)$$

as $x \to 1^-$. *Then*

$$s_n = s + \mathcal{O}\left(\frac{1}{\log(n)}\right)$$

as $n \to \infty$.

Detailed outline of the paper

Following this brief introduction to the relevant background in Tauberian theory, we are now in a position to give a more detailed outline of the main results of the present paper. In recent work [22], the comparatively elementary proofs of Abel's and Tauber's theorems were analysed, and finitary versions of these theorems established. In what follows, we extend this idea to the much more complex Littlewood Tauberian theorem, stating and proving a finitary version of Theorem 2.3 in Section 5 and showing how this can be clearly understood in terms of the Dialectica interpretation of implication. Before doing this, we give a new, Cauchy reformulation of Littlewood's theorem, and present a proof of this reformulation that is inspired by Wielandt's variation [30] of Karamata's method of proof [8].

We then combine our finitization with known bounds on polynomial approximations to derive two remainder theorems for Littlewood's theorem. The first deals with the case where $F(x) \to s$ with some *arbitrary* computable rate of convergence, while the second applies more generally still when $F(x) \to s$ as $x \to 1^-$ but without necessarily having a computable rate of convergence, instead converting a rate of *metastability* of the former to a rate of metastability for $s_n \to s$. Both of these form generalisations of Theorem 2.4, which we show can be derived as a special case. The use of logical methods to obtain Tauberian remainder estimates in this way is completely new.

3 A Cauchy variant of Littlewood's theorem

We start by giving a new presentation of Littlewood's theorem. In line with standard approaches to analysing convergence theorems using proof theoretic methods, we prefer to work with a reformulation of the theorem that minimises its logical complexity. In particular, we seek a version of Theorem 2.3 which, rather than referring directly to the limit s, expresses the relevant convergence properties in an equivalent Cauchy form. In this way we also achieve a fully finitary formulation of the theorem in Section 5, phrased in entirely in terms of finite intervals of stability for $F(x)$ and s_n, and independent of any limit s.

Firstly, the assumption that $F(x) \to s$ as $x \to 1^-$ will be replaced with a natural Cauchy variant which says that for any $\delta > 0$ we have $|F(x) - F(y)| \leq \delta$ for x, y sufficiently close to 1. While it would be natural to then also replace the conclusion $s_n \to s$ as $n \to \infty$ with the standard Cauchy property for convergent sequences, the issue here is Littlewood's theorem doesn't just state that $\{s_n\}$ converges, but that it converges to the *same limit* as $F(x)$ as $x \to 1^-$. Therefore we instead formulate the conclusion as the following Cauchy property: for any $\varepsilon > 0$ we have $|s_n - F(x_m)| \leq \varepsilon$ for sufficiently large m, n, where x_m is some canonical sequence in $[0, 1)$ with $x_m \to 1^-$. From this we can then retrieve that $s_n \to \lim_{x \to 1^-} F(x)$. For reasons that will become clear when we give our proof of the theorem, we choose $x_m := e^{-1/m}$ as our canonical sequence. We present our Cauchy variant of Littlewood's theorem below, and then prove that the original result follows from it.

Definition 3.1. As usual we write $x \in (a,b)$ for $a < x < b$ and $x \in [a,b]$ for $a \leq x \leq b$, and so on. For integers l,m,n we also write $n \in [l,m]$ to denote $l \leq n \leq m$.

Theorem 3.2 (Littlewood's theorem, Cauchy variant)**.** *Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and that*

$$\forall \delta > 0 \exists M \forall x, y \in [e^{-1/M}, 1)(|F(x) - F(y)| \leq \delta)$$

Then we have

$$\forall \varepsilon > 0 \exists N \forall m, n \geq N(|s_n - F(e^{-1/m})| \leq \varepsilon)$$

Before proving this, we show that it can be used to derive the original formulation of Littlewood's theorem:

Proof of Theorem 2.3 from Theorem 3.2. If $a_n = \mathcal{O}(1/n)$ then by definition there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$. Furthermore, if $F(x) \to s$ as $x \to 1^-$ then for any $\delta > 0$ there exists some $0 \leq \mu < 1$ such that

$$x \in [\mu, 1) \implies |F(x) - s| \leq \frac{\delta}{2}$$

Let M be sufficiently large that $e^{-1/M} \geq \mu$. Then $x, y \in [e^{-1/M}, 1)$ implies that

$$|F(x) - F(y)| \leq |F(x) - s| + |s - F(y)| \leq \delta$$

and so we have established the premise of Theorem 3.2. Therefore we can conclude that for any $\varepsilon > 0$ there exists some N such that

$$m, n \geq N \implies |s_n - F(e^{-1/m})| \leq \frac{\varepsilon}{2}$$

But since $F(x) \to s$ as $x \to 1^-$ we can choose $m_0 \geq N$ large enough so that $|F(e^{-1/m_0}) - s| \leq \varepsilon/2$, and then $n \geq N$ implies

$$|s_n - s| \leq |s_n - F(e^{-1/m_0})| + |F(e^{-1/m_0}) - s| \leq \varepsilon$$

and we've proven that $s_n \to s$ as $n \to \infty$. □

We now give a proof of Theorem 3.2, which will be analysed in the next section. This is an adaptation of Wielandt's proof of the original theorem [30] (see also [14, Chapter I.12]). Roughly speaking, this proof strategy, which dates back to Karamata [8], is based on representing the partial sums s_n as step functions, and using integral theory to show that these approach $F(e^{-1/m})$ as $m \to \infty$. Our use of integrals in this way requires us to replace the discontinuous step function with a polynomial approximation to it. That we are able to find such polynomials for arbitrary errors relies on a result from approximation theory, and we begin by stating this in the exact form in which it is needed:

Lemma 3.3. *Define* $\chi : [0,\infty) \to \mathbb{R}$ *to be*

$$\chi(t) := \begin{cases} 1 & \text{if } t \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

Then for any $\varepsilon > 0$ *there exists a polynomial* P *with* $P(0) = 0$ *and* $P(1) = 1$ *satisfying*

$$\int_0^\infty \frac{|\chi(t) - P(e^{-t})|}{t}\, dt < \varepsilon \tag{1}$$

Proof sketch. This is a standard result (cf. [13] or [14, Chapter I.11–12] for full details), and so we just sketch the idea. We first consider the function $f : [0,1] \to \mathbb{R}$ defined by

$$f(t) := \frac{\chi(\log(1/t)) - t}{t(1-t)}$$

which is Lipschitz continuous on both $[0, 1/e)$ and $[1/e, 1]$ and has a single jump discontinuity at $t = 1/e$. It is a well known fact that such functions can be approximated to arbitrary precision by polynomials, so we pick some polynomial p such that

$$\int_0^1 |f(t) - p(t)|\, dt < \varepsilon$$

and then set $P(t) := t + t(1-t)p(t)$, noting that $P(0) = 0$ and $P(1) = 1$. Substituting $t = e^{-u}$ in the above integral we have

$$\int_0^1 |f(t) - p(t)|\, dt = \int_0^\infty \frac{|\chi(u) - P(e^{-u})|}{1 - e^{-u}}\, du$$

and applying the standard inequality $1 + x \leq e^x$ in the form $1 - e^{-u} \leq u$ yields

$$\frac{|\chi(u) - P(e^{-u})|}{u} \leq \frac{|\chi(u) - P(e^{-u})|}{1 - e^{-u}}$$

for $0 < u$, and therefore

$$\int_0^\infty \frac{\chi(u) - P(e^{-u})}{u}\, du \leq \int_0^1 |f(t) - p(t)|\, dt < \varepsilon$$

which completes the proof. □

We are now ready to apply the above lemma to prove our Cauchy variant of Littlewood's theorem.

Proof of Theorem 3.2. Fix $\varepsilon > 0$ for the remainder of the proof, and let $P(t)$ be as in Lemma 3.3 for error $\varepsilon/4C$. We let $a : [0,\infty) \to \mathbb{R}$ be the discontinuous step function corresponding to $\{a_n\}$, defined by $a(t) := a_n$ for $t \in [n, n+1)$, and define

$$I_P(n) := \int_0^\infty a(t) P(e^{-t/n})\, dt$$

Our main strategy is to show that

(i) $|s_n - I_P(n)| \le \varepsilon/2$ for all $n \in \mathbb{N}$ and,

(ii) $|I_P(n) - F(e^{-1/m})| \le \varepsilon/2$ for sufficiently large m, n.

Then putting these together we have

$$|s_n - F(e^{-1/m})| \le |s_n - I_P(n)| + |I_P(n) - F(e^{-1/m})| \le \varepsilon$$

for sufficiently large m, n, and since ε was arbitrary the theorem is proved. We tackle each of these in turn, using the growth condition $n|a_n| \le C$ for (i) and convergence of $F(x)$ as $x \to 1^-$ for (ii).

For (i), we first note that that

$$s_n = \int_0^n a(t)\,dt = \int_0^\infty a(t)\chi(t/n)\,dt$$

and therefore

$$|s_n - I_P(n)| \le \int_0^\infty |a(t)| \cdot |\chi(t/n) - P(e^{-t/n})|\,dt \tag{2}$$

Now for any $n \in \mathbb{N}$ we have $n|a_n| \le C$ and therefore for $n \ge 1$ and $t \in [n, n+1)$:

$$|a(t)| = |a_n| \le \frac{C}{n} \le \frac{2C}{n+1} \le \frac{2C}{t} \tag{3}$$

Similarly, for $t \in (0,1)$, assuming w.l.o.g. that $|a_0| \le C$ (otherwise we can just modify the constant and set $C' := \max\{|a_0|, C\}$), we also have $|a(t)| = |a_0| \le C \le 2C/t$, and thus it follows that $|a(t)| \le 2C/t$ for all $t \in (0,\infty)$. Therefore from (2) we have

$$\begin{aligned} |s_n - I_P(n)| &\le 2C \int_0^\infty \frac{|\chi(t/n) - P(e^{-t/n})|}{t}\,dt \\ &\le 2C \int_0^\infty \frac{|\chi(u) - P(e^{-u})|}{u}\,du < \frac{\varepsilon}{2} \end{aligned} \tag{4}$$

using the substitution $u = t/n$ and the property of P.

To prove (ii), we first note that by Lemma 3.3 we know that $P(0) = 0$ and $P(1) = 1$, which implies that $P(t) = \sum_{i=1}^d c_i t^i$ for some $d \in \mathbb{N}$ and $c_1, \dots, c_d$ with $\sum_{i=1}^d c_i = 1$. Writing out $I_P(n)$ as

$$I_P(n) = \sum_{i=1}^d c_i \int_0^\infty a(t) e^{-it/n}\,dt \tag{5}$$

and observing that for any $\alpha > 0$ we have

$$\begin{aligned} \int_0^\infty a(t)e^{-\alpha t}\,dt &= \sum_{k=0}^\infty a_k \int_k^{k+1} e^{-\alpha t}\,dt \\ &= \sum_{k=0}^\infty a_k e^{-\alpha k} \int_0^1 e^{-\alpha t}\,dt = F(e^{-\alpha}) \int_0^1 e^{-\alpha t}\,dt \end{aligned} \tag{6}$$

then combining (5) and (6) we have

$$I_P(n) = \sum_{i=1}^{d} c_i F(e^{-i/n}) \int_0^1 e^{-it/n}\, dt \tag{7}$$

Using now that $\sum_{i=1}^{d} c_i = 1$ and thus

$$F(e^{-1/m}) = \sum_{i=1}^{d} c_i F(e^{-1/m})$$

it follows from (7) that

$$\begin{aligned}
&|I_P(n) - F(e^{-1/m})| \\
&= \left| \sum_{i=1}^{d} c_i \left(F(e^{-i/n}) \int_0^1 e^{-it/n}\, dt - F(e^{-1/m}) \right) \right| \\
&\leq \sum_{i=1}^{d} |c_i| \cdot |F(e^{-i/n}) \int_0^1 e^{-it/n}\, dt - F(e^{-1/m})| \\
&\leq \sum_{i=1}^{d} |c_i| \cdot \left(F(e^{-1/n}) \left(\int_0^1 e^{-it/n}\, dt - 1 \right) + (F(e^{-i/n}) - F(e^{-1/m})) \right) \\
&\leq \sum_{i=1}^{d} |c_i| \cdot \left(L \left| \int_0^1 e^{-it/n}\, dt - 1 \right| + |F(e^{-i/n}) - F(e^{-1/m})| \right)
\end{aligned} \tag{8}$$

where for the last step we use that $|F(x)|$ must be bounded above by some $L > 0$ as $x \to 1^-$. Now for any fixed $i = 1, \ldots, d$ we have

$$\lim_{n\to\infty} \int_0^1 e^{-it/n}\, dt = 1$$

and by our Cauchy convergence condition $|F(e^{-i/n}) - F(e^{-1/m})| \to 0$ as $m, n \to \infty$. It therefore follows from (8) that we can choose m, n sufficiently large so that $|I_P(n) - F(e^{-1/m})| \leq \varepsilon/2$. This proves (ii), and we're done □

4 The Dialectica interpretation

Having motivated and presented Littlewood's Tauberian theorem, in a Cauchy form whose simplified logical structure makes it easier to interpret, we now briefly outline the logical technique used to finitize the theorem: Gödel's Dialectica interpretation. It is important to note that this paper is not a rigorous study of the Dialectica itself – rather the interpretation acts as a guide in how to correctly formulate our finitary version of Littlewood's theorem, along with the subsequent remainder theorems. As such, exact details of the interpretation and the surrounding theory are not important here, and a prior familiarity with the interpretation not required to appreciate our quantitative theorems. With this in mind, we restrict our attention to key features of the interpretation that play a role in what follows. For detailed background on the Dialectica interpretation see e.g. [1] or [10].

4.1 The basic interpretation, and how it is used in this paper

Numerous different variants of the Dialectica interpretation have been explored over the years, ranging from Gödel's original interpretation to specialised versions used in proof mining [10] or the theory of programming languages [19, 20]. In its standard form the Dialectica assigns to any formula A in some formal (semi) intuitionistic theory P_I a logically equivalent formula of the form $A^D = \exists x \forall y A_D(x,y)$ in some higher-order variant P_ω of that theory, where

- $A_D(x,y)$ is 'computationally neutral' (typically quantifier free or decidable),
- the free variables of $\exists x \forall y A_D(x,y)$ are the same as those of A,
- x and y are (potentially empty) tuples of terms in all finite types, where these types depend on A.

To be more precise, $A_D(x,y)$ is defined by induction over the logical structure of A as follows:

$$\begin{aligned}
A^D &:= A \quad \text{if } A \text{ is computationally neutral} \\
(A \wedge B)^D &:= \exists x, u \forall y, v (A_D(x,y) \wedge B_D(u,v)) \\
(A \vee B)^D &:= \exists b, x, u \forall y, v (A_D(x,y) \vee_b B_D(u,v)) \\
(A \implies B)^D &:= \exists f, g \forall x, v (A_D(x, gxv) \implies B_D(fx, v)) \\
(\exists z A[z])^D &:= \exists z, x \forall u \, A_D[z](x,u) \\
(\forall z A[z])^D &:= \exists f \forall z, u \, A_D[z](fz, y)
\end{aligned}$$

where in the interpretation of disjunction b is a boolean or natural number and $P \vee_b Q$ is shorthand for

$$(b = 0 \implies P) \wedge (b \neq 0 \implies Q)$$

The most fundamental results concerning the Dialectica interpretation are *soundness theorems*, which guarantee that whenever A is provable in P_I, a computable witnessing term for A^D can be extracted from the proof:

Intuitionistic soundness theorem: If $P_I \vdash A$ then we can extract, from the proof of A, some term t in P_ω such that $P_\omega \vdash \forall y A_D(t,y)$.

The original soundness theorem due to Gödel instantiates P_I as Heyting arithmetic and P_ω as System T, though many more soundness theorems have been developed since. In the case of classical theories P_C, it is not always possible to extract computable terms that witness the Dialectica interpretation of formulas. However, here one can instead precompose the Dialectica with a negative translation A^N that embeds P_C into its intuitionistic variant P_I:

Classical soundness theorem: If $P_C \vdash A$ then we can extract, from the proof of A, some term t in P_ω such that $P_\omega \vdash \forall y (A^N)_D(t, y)$.

The value of the soundness theorems for applied proof theory lies primarily in the fact that they set out conditions under which it is theoretically possible to extract computational content from proofs, determine the precise shape that this computational information should take, and provide a recipe for extracting realizing terms from formalized versions of those proofs. However, outside of formal program synthesis (to obtain verified programs via a proof assistant, for example), the soundness theorems are rarely applied step-by-step to fully formalized proofs. Rather, the interpretation is used in an informal way to guide the extraction process, in conjunction with ordinary mathematical intuition.

More concretely, in our case the Dialectica interpretation dictates the way in which we formulate our finitary Tauberian theorem (Theorem 5.3 below), and (in its classical variant) shows us how to obtain meaningful remainder theorems even when no computable rates of convergence exist in Section 6.2. However, our quantitative results are all proven 'by-hand' and without the rigorous application of logical methods. So while the soundness proof for the Dialectica interpretation certainly helped guide our analysis of the proof of Theorem 3.2, we do not include any formal details. We instead prefer to present our proof of the finitary theorem in an ordinary mathematical style. It should, however, be clear that the proof of Theorem 5.3 directly mirrors and forms a computational analogue to that of Theorem 3.2.

4.2 The interpretation of implication

The characterising feature of the Dialectica interpretation that sets it apart from similar proof interpretations such as modified realizability is its interpretation of implication. It is this that informs our computational interpretation of Littlewood's theorem, and so it is important to give some insight into its meaning. In interpreting implication, we are required to choose a Skolemisation of the formula:

$$\exists x \forall y A_D(x, y) \implies \exists u \forall v B_D(u, v)$$

There are various options available, but the Dialectica selects the 'least nonconstructive' of these, which turns out to be:

$$\forall x \exists u \forall v \exists y (A_D(x, y) \implies B_D(u, v)) \tag{9}$$

(for the reasoning behind this choice see e.g. [10, pp. 128–129]). Bringing the existential quantifiers $\exists u$ and $\exists y$ to the front as functions gives us exactly $(A \implies B)^D$. Happily, the formula (9) can be given an elegant reading in terms of game semantics, as a game between $\exists$loise and $\forall$belard where $\exists$loise seeks to prove $A^D \implies B^D$ while $\forall$belard attempts to contradict $\exists$loise by proving $A^D \wedge \neg B^D$:

1. $\forall$belard begins by proposing a witness x such that $\forall y A_D(x,y)$ holds, with the aim of disproving B^D.

2. $\exists$loise responds by proposing a witness u such that $\forall v B_D(u,v)$ is true.

3. $\forall$belard now tried to contradict $\exists$loise's witness for B^D by proposing a counterexample v such that $\neg B_D(u,v)$.

4. $\exists$loise responds by contradicting $\forall$belard's original claim that A^D is true by providing a counterexample y such that $\neg A^D(x,y)$.

Thus a pair of function f,g that witness $(A \implies B)^D$ represent nothing other than a winning strategy for $\exists$loise in the above game, and this intuitive reading of the Dialectica interpretation in this case is outlined in its abstract form here as it is important in understanding our game-theoretic reading of Littlewood's theorem in Section 5.3 below.

5 A finitary Littlewood Tauberian theorem

We now motivate and present our computational interpretation of Theorem 3.2. We characterise this as a 'finitary' Tauberian theorem because its statement only refers to finite parts of the input data: Rather than asking that $F(x) \to s$ as $x \to 1^-$ for some limit s, we only require that $|F(x) - F(y)|$ is sufficiently small within some fixed range $[l,r] \subset [0,1)$ (see also Remark 5.4 below which further justifies how this statement can be viewed as finitary). Similarly, the growth condition is replaced by an assumption that $n|a_n| \leq C$ for $n \leq p$ for some suitable p. Finally, our conclusion is also finitary in nature: instead of proving that $s_n \to s$ as $n \to \infty$, we establish that $|s_n - F(e^{-1/m})|$ is sufficiently small within some finite range $N \leq m,n \leq k$. In this sense, our notion of finitary coincides with what T. Tao regards as the 'hard' version of a soft analytical statement [25]. Nevertheless, despite only referring to finite parts of the data, our finitary theorem is equivalent to Theorem 3.2, and therefore also Littlewood's original formulation of the 'big-$\mathcal{O}$' Tauberian theorem, in the same way that the Dialectica interpretation of implication (9) is equivalent to $A \implies B$. We discuss this further in Section 5.4.

5.1 The logical structure of Littlewood's theorem

In order to make the quantifier structure of Theorem 3.2 precise, we first introduce some predicate annotations.

Definition 5.1. Fixing some $\{a_n\}$ and $C > 0$, the formulas $A(p)$, $B(\delta, M, l)$ and $D(\varepsilon, N, k)$ are defined as follows:

$$\begin{aligned} A(p) &:= \forall n \leq p (n|a_n| \leq C) \\ B(\delta, M, l) &:= \forall x, y \in [e^{-1/M}, e^{-1/(M+l)}](|F(x) - F(y)| \leq \delta) \\ D(\varepsilon, N, k) &:= \forall m, n \in [N, N+k](|s_n - F(e^{-1/m})| \leq \varepsilon) \end{aligned}$$

The overall structure of Littlewood's theorem (and indeed Tauberian theorems in general) is an implication of the form

$$\text{(convergence)} \wedge \text{(growth condition)} \implies \text{(convergence)}$$

Technically, the quantifier structure of such an implication is rather complex, as each convergence property is a $\forall\exists\forall$ statement. However, just as in the prior analysis of simple Tauberian theorems in [22], an inspection of the proof of Theorem 3.2 reveals that we have in fact proven something stronger: For any $\varepsilon > 0$ there is a concrete δ dependent only on ε, namely $\delta_\varepsilon := \varepsilon/4\sum_{i=1}^{d}|c_i|$, such that $|F(x) - F(y)| \leq \delta_\varepsilon$ for all x, y sufficiently close to 1 implies that $|s_n - F(e^{-1/m})| \leq \varepsilon$ for all m, n sufficiently large (we do not justify this in detail at this point as this will be implicitly proven in Theorem 5.3 below). This is clearly a quantitative strengthening of the theorem, and renders it of the form

$$\forall\varepsilon[\forall p\, A(p) \wedge \exists M\forall l\, B(\delta_\varepsilon, M, l) \implies \exists N\forall k\, D(\varepsilon, N, k)] \tag{10}$$

Now, taking the Dialectica interpretation of the premise of (10) gives us

$$\forall\varepsilon[\exists M\forall p, l\,(A(p) \wedge B(\delta_\varepsilon, M, l)) \implies \exists N\forall k\, D(\varepsilon, N, k)]$$

and the Dialectica interpretation of the above implication as in (9) yields the following as a final Skolemisation of the Littlewood Tauberian theorem in its Cauchy formulation:

$$\forall\varepsilon, M\exists N\forall k\exists p, l[A(p) \wedge B(\delta_\varepsilon, M, l) \implies D(\varepsilon, N, k)] \tag{11}$$

5.2 The finitary theorem

Our finitary version of Littlewood's theorem corresponds to the Dialectica interpretation of Theorem 3.2 in the form (11). More importantly, we analyse the proof of Theorem 3.2 to provide concrete bounds on N in terms of ε and M and on p and l in terms of ε, M and k, parametrised by the big-$\mathcal{O}$ bound $C > 0$, together with a uniform bound $a > 0$ for the sequence $\{|a_n|\}$, the latter being necessary since we can no longer derive boundedness of $\{|a_n|\}$ from the growth condition, as we now only assume a finitary version of $a_n = \mathcal{O}(1/n)$.

Theorem 3.2 relies crucially on the existence of approximating polynomials as set out in Lemma 3.3. The first step in our analysis is to examine the proof of Theorem 3.2 and identify exactly which numerical aspects of Lemma 3.3 are required to obtain witnesses for (11). It turns out that it is sufficient to have, for any $\varepsilon > 0$, bounds on both the degree d and the sum of the magnitude of the coefficients $\sum_{i=1}^{d}|c_i|$ for a polynomial $P(x) = c_1x + \ldots + c_dx^d$ satisfying (1). We can express this more precisely as follows:

Definition 5.2. Let χ be defined as in Lemma 3.3. We define a *modulus of polynomial approximation for* χ to be any function $\Omega : (0, \infty) \to (0, \infty) \times (0, \infty)$

such that for any $\varepsilon > 0$ there exists a polynomial $P(x) = \sum_{i=1}^{d} c_i x^i$ satisfying Lemma 3.3 with error ε, such that

$$d \leq \Omega_0(\varepsilon) \quad \text{and} \quad \sum_{i=1}^{d} |c_i| \leq \Omega_1(\varepsilon)$$

where $\Omega(\varepsilon) = (\Omega_0(\varepsilon), \Omega_1(\varepsilon))$.

We now state and prove our finitary Tauberian theorem in terms of some generic modulus of polynomial approximation Ω, before supplying a concrete modulus in Lemma 5.5.

Theorem 5.3 (Finitary Tauberian theorem)**.** *Suppose that $C > 0$ is some real number, $a > 0$ is a bound on $\{|a_n|\}$, and Ω is a modulus of polynomial approximation for χ. Fix $\varepsilon > 0$ and let b, v and δ be defined by*

$$(b, v) := \Omega\left(\frac{\varepsilon}{8C}\right)$$
$$\delta := \frac{\varepsilon}{4v}$$

Given $M \in \mathbb{N}$, define $N \in \mathbb{N}$ by

$$N := b \cdot \max\left\{\left\lceil \frac{L}{\delta} \right\rceil, M\right\} \quad \textit{for} \quad L := \frac{a}{1 - e^{-1/M}} + \delta$$

Finally, given $k \in \mathbb{N}$ define $p, l \in \mathbb{N}$ by

$$l := N + k - M$$
$$p := (N + k) \cdot \max\left\{\left\lceil \log\left(\frac{a(N + k)}{\delta}\right)\right\rceil, 1\right\}$$

Then from

$$n|a_n| \leq C \quad \textit{for all } n \leq p \tag{12}$$

and

$$|F(x) - F(y)| \leq \delta \quad \textit{for all } x, y \in [e^{-1/M}, e^{-1/(M+l)}] \tag{13}$$

it follows that

$$|s_n - F(e^{-1/m})| \leq \varepsilon \quad \textit{for all } m, n \in [N, N + k]$$

Remark 5.4. We can actually say something more precise instead of (13): We in fact only need $|F(x) - F(y)| \leq \delta$ for all $x, y := e^{-i/n}, e^{-1/m}$ for $i \leq b$ and $m, n/i \in [M, M + l]$, and so in reality the assumption only needs to hold for a finite number of points. In addition to this, we note that even though the computation of $F(x)$ technically requires the whole of the sequence $\{a_n\}$, within any closed interval $[l, r] \in [0, 1)$ the power series has a uniform rate of convergence, namely

$$\sum_{i=n}^{\infty} |a_i x^i| \leq \frac{a x^n}{1 - x} \leq \frac{a r^n}{1 - r}$$

and so we could also adapt (13) so that it only refers to finite sums of the form $\sum_{i=0}^n a_i x^i$ for sufficiently large n depending on M, l and δ. However, we leave our slightly simpler formulation as it is, as it is mathematically cleaner and sufficient for deriving remainder theorems in the next section.

Our proof of Theorem 5.3 forms a constructive analysis of the proof of Theorem 3.2. In the original proof, we fix $\varepsilon > 0$, and letting $P(t)$ be a suitable polynomial approximation to χ, the proof splits into two main branches, namely:

$$\forall p\, A(p) \implies \forall n(|s_n - I_P(n)| \leq \varepsilon/2)$$
$$\exists M \forall l\, B(\delta_\varepsilon, M, l) \implies \exists N \forall m, n \geq N(|I_P(n) - F(e^{-1/m})| \leq \varepsilon/2)$$

where $A(p)$ and $B(\delta, M, l)$ are as in Definition 5.1. We analyse each branch in turn, extracting our witness for p from the former, and our witness for l in the latter. Our analysis broadly follows the soundness theorem for the Dialectica, but we use several tricks along the way. We note that this combination of formal logical methods with ordinary mathematical intuition is standard in applied proof theory, and key to its success.

Proof of Theorem 5.3. We follow the structure of the proof of Theorem 3.2, but backwards, carrying out precise numerical calculations along the way. We suppose for contradiction that

$$\varepsilon < |s_n - F(e^{-1/m})|$$

for some $m, n \in [N, N+k]$. Now let $P(x) = c_d x^d + \ldots + c_1 x$ be the polynomial that satisfies Lemma 3.3 for error $\varepsilon/8C$, noting that $d \leq b$ and $\sum_{i=1}^d |c_i| \leq v$. Define $a : [0, \infty) \to \mathbb{R}$ and I_P as in the proof of Theorem 3.2. Then we have

$$\varepsilon < |s_n - I_P(n)| + |I_P(n) - F(e^{-1/m})|$$

and therefore either

(i) $\varepsilon/2 < |s_n - I_P(n)|$ for some $n \leq N + k$ or,

(ii) $\varepsilon/2 < |I_P(n) - F(e^{-1/m})|$ for some $m, n \in [N, N+k]$.

We treat each possibility in turn, each leading to a contradiction in (12) or (13) respectively.

Case (i): $\varepsilon/2 < |s_n - I_P(n)|$ for $n \leq N + k$. We first observe that $p \geq N + k$ and therefore for any $t \geq p$ we have

$$\frac{t}{n} \geq \frac{t}{N+k} \geq \frac{p}{N+k} \geq 1$$

and thus $\chi(t/n) = 0$ for $t \in [p, \infty)$. Therefore using that $a_n \leq a$ for all $n \in \mathbb{N}$ we have

$$\int_p^\infty |a(t)| \cdot |\chi(t/n) - P(e^{-t/n})|\, dt \leq a \int_p^\infty |P(e^{-t/n})|\, dt$$

Now, for any $x \in [0, 1)$ we have

$$|P(x)| \leq \sum_{i=1}^{d} |c_i| x^i \leq x \cdot \sum_{i=1}^{d} |c_i| \leq xv$$

and therefore

$$a \int_p^\infty |P(e^{-t/n})| \, dt \leq av \int_p^\infty e^{-t/n} \, dt = avne^{-p/n} \tag{14}$$

Using again that $n \leq N + k$ and thus

$$p \geq n \cdot \log \left(\frac{an}{\delta} \right)$$

we obtain $\delta/an \geq e^{-p/n}$ and therefore

$$avne^{-p/n} \leq v\delta = \frac{\varepsilon}{4} \tag{15}$$

Using (2) together with (14) and (15) we have

$$\frac{\varepsilon}{2} < |s_n - I_P(n)| \leq \int_0^p |a(t)| \cdot |\chi(t/n) - P(e^{-t/n})| \, dt + \frac{\varepsilon}{4}$$

and therefore

$$\frac{\varepsilon}{4} < \int_0^p |a(t)| \cdot |\chi(t/n) - P(e^{-t/n})| \, dt$$

But by (12), and using a similar argument to that in the proof of Theorem 3.2, we have $|a(t)| \leq 2C/t$ for all $t \in (0, p]$ and thus using (4):

$$\begin{aligned}
\frac{\varepsilon}{4} &< 2C \int_0^p \frac{|\chi(t/n) - P(e^{-t/n})|}{t} \, dt \\
&\leq 2C \int_0^\infty \frac{|\chi(t/n) - P(e^{-t/n})|}{t} \, dt \\
&< 2C \cdot \frac{\varepsilon}{8C} = \frac{\varepsilon}{4}
\end{aligned}$$

contradicting the construction of $P(t)$.

Case (ii): $\varepsilon/2 < |I_P(n) - F(e^{-1/m})|$ for $m, n \in [N, N + k]$. First of all, define

$$\epsilon_\alpha := 1 - \int_0^1 e^{-\alpha t} \, dt$$

For $\alpha > 0$ we have

$$e^{-\alpha} \leq \int_0^1 e^{-\alpha t} \, dt \leq 1$$

and therefore

$$0 \leq \epsilon_\alpha \leq 1 - e^{-\alpha} \leq \alpha \tag{16}$$

Now, analogously to (8) we see that

$$\begin{aligned}\frac{\varepsilon}{2} &< |I_P(n) - F(e^{-1/m})| \\ &\leq \sum_{i=1}^{d} |c_i| \cdot |F(e^{-i/n})(1-\epsilon_{i/n}) - F(e^{-1/m})| \\ &\leq \sum_{i=1}^{d} |c_i| \cdot (\epsilon_{i/n}|F(e^{-i/n})| + |F(e^{-i/n}) - F(e^{-1/m})|)\end{aligned} \tag{17}$$

We first aim to bound the term $\epsilon_{i/n}|F(e^{-i/n})|$ uniformly for $i = 1, \ldots, d$. Using $bM \leq N \leq n \leq N + k$ we have

$$\frac{1}{M+l} = \frac{1}{N+k} \leq \frac{i}{n} \leq \frac{b}{N} \leq \frac{b}{bM} = \frac{1}{M} \tag{18}$$

So to bound $|F(e^{-i/n})|$ we observe from (18) that $e^{-i/n} \in [e^{-1/M}, e^{-1/(M+l)}]$ and thus by condition (13)

$$|F(e^{-i/n})| \leq |F(e^{-1/M})| + \delta \leq \frac{a}{1-e^{-1/M}} + \delta = L \tag{19}$$

where for the second inequality we have used that

$$F(x) \leq a \sum_{i=0}^{\infty} x^i = \frac{a}{1-x}$$

for $x \in [0, 1)$. Therefore from (16), (18) and (19) we have

$$\epsilon_{i/n}|F(e^{-i/n})| \leq \frac{i}{n} \cdot |F(e^{-i/n})| \leq \frac{b}{N} \cdot L \leq \delta \tag{20}$$

where the final inequality follows from the definition of N. Finally, analogously to (18) we have $1/(M+l) \leq 1/m \leq 1/M$ and thus $e^{-1/m} \in [e^{-1/M}, e^{-1/(M+l)}]$, and so from (13) we have

$$|F(e^{-i/n}) - F(e^{-1/m})| \leq \delta \tag{21}$$

Finally, putting together (17), (20) and (21) we have

$$\frac{\varepsilon}{2} < \sum_{i=1}^{d} |c_i| \cdot (\delta + \delta) \leq 2v\delta = \frac{\varepsilon}{2}$$

a contradiction. This completes the proof. □

We conclude by giving a concrete modulus of polynomial approximation, which can be obtained using standard results from approximation theory, in particular Korevaar [13].

Lemma 5.5. *There are constants $A, B > 0$ such that*

$$\Omega(\varepsilon) := \left(\frac{A}{\varepsilon}, B^{1/\varepsilon}\right)$$

is a modulus of polynomial approximation for χ and $\varepsilon \in (0,1]$ (note that we can just set $\Omega(\varepsilon) := \Omega(1)$ for $\varepsilon \geq 1$).

Proof. We adapt Korevaar [13] in a straightforward way. Let $f : [0,1] \to \mathbb{R}$ be any function that is continuous except for a finite number of jump discontinuities, and such that there exists a constant a such that $|f(x) - f(y)| \leq a|x - y|$ for x, y in any subinterval on which f is continuous. Then it can be shown (cf. [13, Theorem 4.1]) that there exist constants H_1 and H_2 such that for any positive integer n there exists a polynomial $p_n(x) = \alpha_n x^n + \ldots + \alpha_1 x + \alpha_0$ of degree n and with $|\alpha_i| \leq H_2 3^n$ for all $i = 0, 1, \ldots, n$ such that

$$\int_0^1 |f(t) - p_n(t)|\, dt < \frac{H_1}{n+1}$$

Now let f be as in the proof of Lemma 5.5, and H_1, H_2 be the constants corresponding to this function. Then, just as in that proof, we set $P_n(t) = t + t(1-t)p_n(t)$ and have

$$\int_0^\infty \frac{|\chi(t) - P_n(e^{-t})|}{t}\, dt < \frac{H_1}{n+1}$$

noting that $P_n(t) = c_{n+2}t^{n+2} + \ldots c_1 t$ has degree $n+2$, and that

$$\sum_{i=1}^{n+2} |c_i| \leq 2\sum_{i=0}^{n} |\alpha_i| + 1 \leq 2(n+1)H_2 3^n + 1 \leq H_3 6^n$$

for suitable $H_3 \geq 1$ depending on H_2. Now, for any $\varepsilon > 0$ it is clear that setting $P(t) := P_n(t)$ for $n := \lceil H_1/\varepsilon \rceil$ we have

$$\int_0^\infty \frac{|\chi(t) - P(e^{-t})|}{t}\, dt < \varepsilon$$

and that for $\varepsilon \in (0,1]$:

$$\deg(P) = \lceil H_1/\varepsilon \rceil + 2 \leq \frac{H_1 + 3}{\varepsilon}$$

and so we can set $A := H_1 + 3$. Finally, for $\varepsilon \in (0,1]$ we have

$$\sum_{i=1}^{n+2} |c_i| \leq H_3 6^{\lceil H_1/\varepsilon \rceil} \leq B^{1/\varepsilon}$$

for a suitable choice of B, depending on H_1 and H_3. □

5.3 A game semantics for Littlewood's theorem

Though the statement of our finitary theorem is somewhat complex, using the game-theoretic narrative from Section 4.2 we can give Theorem 5.3 a slightly more dynamic character, as setting out a winning strategy in a game corresponding to the Littlewood Tauberian theorem. Here, ∃loise sets out to foil ∀belard's attempt to disprove Littlewood's theorem by showing that $a_n = \mathcal{O}(1/n)$, $F(x) \to s$ and $s_n \not\to s$ all hold together:

1. ∀belard starts by picking some $\varepsilon > 0$, assuming that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and proposing some $M \in \mathbb{N}$ such that $|F(x) - F(y)| \leq \delta$ for all $x, y \in [e^{-1/M}, 1)$. The aim is to show that it is now not the case that $|s_n - F(e^{-1/m})| \leq \varepsilon$ for sufficiently large m, n.

2. ∃loise responds by putting forward an $N \in \mathbb{N}$ such that $|s_n - F(e^{-1/m})| \leq \varepsilon$ for all $m, n \geq N$.

3. ∀belard rejects this by attempting to find a counterexample to the last move, playing $k \in \mathbb{N}$ and claiming that $\varepsilon < |s_n - F(e^{-1/m})|$ for some $n, m \in [N, N + k]$.

4. If ∀belard's attempt worked, then ∃loise responds by producing a pair $l, p \in \mathbb{N}$ which demonstrate that one of ∀belard's original assumptions was false:

   - either $C < n|a_n|$ for some $n \leq p$, or
   - $\delta < |F(x) - F(y)|$ for some $x, y \in [e^{-1/M}, e^{-1/(M+l)}]$.

A winning strategy for ∃loise constitutes a proof of the Littlewood Tauberian theorem, and such a winning strategy is provided by Theorem 5.3 in presenting bounds for winning moves for ∃loise in terms of any play from ∀belard .

5.4 Obtaining the original Tauberian theorem from its finitization

We conclude this section by remarking that Theorem 5.3 is completely equivalent to the standard Tauberian theorem. This follows directly from the general fact that a formula is logically equivalent to its Dialectica interpretation, but for completeness we show how the original variant follows from the finitary one, noting that there is no content to the proof beyond standard predicate logic.

Proof of Theorem 3.2 from Theorem 5.3. Suppose for contradiction that the conditions of Theorem 3.2 holds but the conclusion is false. This means that there exists some $\varepsilon > 0$ such that

$$\forall N \exists m, n \geq N (|s_n - F(e^{-1/m})| > \varepsilon) \tag{22}$$

Let $\delta > 0$ be defined as in Theorem 5.3 relative to this ε, and pick M such that

$$\forall x, y \in [e^{-1/M}, 1)(|F(x) - F(y)| \leq \delta)$$

Now let $N \in \mathbb{N}$ be defined as in Theorem 5.3 relative to ε and M. By (22) there exists $m, n \geq N$ such that $|s_n - F(e^{-1/m})| > \varepsilon$. Finally, let l and p be defined as in Theorem 5.3 relative to ε, N and $k := \max\{m, n\}$. Then (12) and (13) hold, but the conclusion of Theorem 5.3 fails. □

6 Remainder theorems

The final contribution in this paper is to use our finitization of Littlewood's theorem to (re)derive a series of general "remainder theorems". Remainder theorems have been widely studied in Tauberian theory, and the standard estimate for Littlewood's theorem (Theorem 2.4) has been broadly generalised to more powerful Tauberian theorems (cf. [14, Chapter VII]). Our contribution in the present paper does not seek to improve or replace existing remainder theorems (though whether there are situations where proof theoretic methods are indeed able to produce improved remainder theorems in Tauberian theory is a fastinating open question), but generalises them in a slightly different direction:

- We show that standard remainder theorems, and in particular Theorem 2.4, can be rederived in a 'proof-theoretic' way from, and form an instance of, our finitization of Tauber's theorem.
- In the case where no rates of convergence exists, we can instead use our finitization to produce a 'metastable' remainder theorem.

For us, a remainder theorem will be a quantitative form of the Tauberian theorem which specializes the finitary theorem of the previous section in the following ways:

(i) We assume that $a_n = \mathcal{O}(1/n)$ rather than reformulating the growth condition in a finitary way,

(ii) For simplicity, we take as an additional parameter some $L > 0$ that bounds $|F(x)|$ on $x \in [0, 1)$ (which exists by the assumption that $F(x) \to s$),

(ii) We convert some quantitative measure of the convergence speed of $F(x) \to s$ as $x \to 1^-$ to a measure of the convergence speed of $s_n \to s$ as $n \to \infty$.

These results will form generalisations of Theorem 2.4, and are possible due to the analysis of the proof of Theorem 3.2 and the resulting witnesses of its Dialectica interpretation given in Theorem 5.3. A key step towards these remainder theorems is the following corollary of Theorem 5.3, which we gain through our assumptions (i) and (ii) above:

Corollary 6.1. *Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and let $L > 0$ be a bound on $|F(x)|$ for $x \in [0, 1)$. Define*

$\alpha : (0,\infty) \to (0,\infty)$, $\beta : \mathbb{N}\times\mathbb{N} \to \mathbb{N}$ *and* $\gamma : (0,\infty)\times\mathbb{N}\times\mathbb{N} \to \mathbb{N}$ *as follows:*

$$\begin{aligned}\alpha(\varepsilon) &:= \frac{\varepsilon}{4\Omega_1(\varepsilon/8C)}\\ \beta(\varepsilon, M) &:= \Omega_0\left(\varepsilon/8C\right)\cdot \max\left\{\left\lceil \frac{L}{\alpha(\varepsilon)}\right\rceil, M\right\}\\ \gamma(\varepsilon, M, k) &:= \beta(\varepsilon, M) + k - M\end{aligned}$$

for Ω *as given in Lemma 5.5. Then we have*

$$\forall \varepsilon, M, k[B(\alpha(\varepsilon), M, \gamma(\varepsilon, M, k)) \implies D(\varepsilon, \beta(\varepsilon, M), k)] \tag{23}$$

where the formulas $B(\delta, M, l)$ *and* $D(\varepsilon, N, k)$ *are defined as in Section 5.1.*

Proof. Directly from Theorem 5.3, noting that $\delta = \alpha(\varepsilon)$, $N = \beta(\varepsilon, M)$ and $l = \gamma(\varepsilon, M, k)$, with the only difference that in Theorem 5.3 we have $L = a/(1 - e^{-1/M}) + \delta$. However, its only role there is to act as a bound for $|F(x)|$ (cf. (19)), and so can be replaced by a general bound on $|F(x)|$. Note that the premise (12) trivially holds if we just assume that $n|a_n| \leq C$ for all $n \in \mathbb{N}$ as we do here, and so p plays no role. □

Letting P stand for the statement that $F(x)$ converges as $x \to 1^-$, and Q the statement that s_n converges as $n \to \infty$, (23) is essentially a Dialectica interpretation of $P \implies Q$. Our remainder theorems take the form of a computational interpretation of modus ponens:

$$\frac{P^* \qquad (P \implies Q)^D}{Q^*}$$

where P^* and Q^* are suitable computational interpretations of P and Q, specifically either direct rates of convergence or rates of metastability. Both of these concepts will be carefully motivated and defined below.

6.1 A remainder theorem for rates of convergence

The canonical remainder estimate for Littlewood's theorem set out as Theorem 2.4 says that whenever $F(x)$ converges with exponential rate, then s_n converges with inverse logarithmic rate. This is a special example of a general phenomenon, which we make precise here, whereby we can convert *any* computable rate of convergence for $F(x)$ into a corresponding computable rate of convergence for s_n. We now state and prove this remainder theorem it using our finitary Tauberian theorem in the form of Corollary 6.1, and then rederive Theorem 2.4 as a special case.

Definition 6.2. We define a rate of convergence for $F(x)$ as $x \to 1^-$ to be any function $\phi : (0,\infty) \to \mathbb{N}$ satisfying

$$\forall \delta > 0 \exists M \leq \phi(\delta) \forall x, y \in [e^{-1/M}, 1)(|F(x) - F(y)| \leq \delta)$$

Similarly, a rate of convergence for $s_n \to \lim_{x\to 1^-} F(x)$ as $n \to \infty$ is defined to be any function $\psi : (0,\infty) \to \mathbb{N}$ satisfying

$$\forall \varepsilon > 0 \exists N \leq \psi(\varepsilon) \forall m, n \geq N (|s_n - F(e^{-1/m})| \leq \varepsilon)$$

We say that a rate of convergence is *computable* if, when restricted to rational inputs $\varepsilon \in (0,\infty)$, it forms a computable function in the usual sense.

Theorem 6.3 (First remainder theorem)**.** *Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and let $L > 0$ be a bound on $|F(x)|$ for $x \in [0,1)$. Suppose that there exists a computable rate of convergence ϕ for $F(x)$ as $x \to 1^-$. Then a computable rate of convergence for $s_n \to \lim_{x\to 1^-} F(x)$ is given by*

$$\psi(\varepsilon) := \beta(\varepsilon, \phi(\alpha(\varepsilon)))$$

where α and β are defined as in Corollary 6.1.

Proof. Letting $B(\delta, M, l)$ and $D(\varepsilon, N, k)$ be defined as in Section 5.1, ϕ being a rate of convergence for $F(x)$ as $x \to 1^-$ is equivalent to the formula

$$\forall \delta > 0 \exists M \leq \phi(\delta) \forall l \, B(\delta, M, l)$$

In particular, for any $\varepsilon > 0$ and $k \in \mathbb{N}$ there exists $M \leq \phi(\alpha(\varepsilon))$ such that

$$\forall k \, B(\alpha(\varepsilon), M, \gamma(\varepsilon, M, k))$$

where γ is as in Corollary 6.1. Therefore from (23) we see that for $N := \beta(\varepsilon, M)$ we have $\forall k \, D(\varepsilon, N, k)$ or equivalently

$$\forall m, n \geq N (|s_n - F(e^{-1/m})| \leq \varepsilon).$$

Finally, we observe that β is monotone in its second argument and thus

$$N = \beta(\varepsilon, M) \leq \beta(\varepsilon, \phi(\alpha(\varepsilon))) = \psi(\varepsilon)$$

and since $\varepsilon > 0$ is arbitrary we have shown that ψ is a rate of convergence for $s_n \to \lim_{x\to 1^-} F(x)$. We finally note that Ω, α, β and γ are all computable, so ψ is also computable. □

We can now give a concrete instance of this remainder theorem that corresponds directly to Theorem 2.4:

Corollary 6.4. *Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and let $L > 0$ be a bound on $|F(x)|$ for $x \in [0,1)$. Suppose that $F(x)$ converges with rate of convergence ϕ where*

$$\phi(\delta) \leq a\delta^{-b}$$

for some $a, b > 0$. Then there exists a rate of convergence ψ for $s_n \to \lim_{x\to 1^-} F(x)$ with

$$\psi(\varepsilon) \leq K^{1/\varepsilon}$$

for $\varepsilon \in (0,1]$, where K is a suitable constant depending on C, L, a and b.

Proof. By our Theorem 6.3 and using the definition of Ω_0 for $\varepsilon \in (0,1]$ from Lemma 5.5 we have

$$\psi(\varepsilon) = \beta(\varepsilon, \phi(\alpha(\varepsilon))) \leq \beta\left(\varepsilon, \frac{a}{\alpha(\varepsilon)^b}\right)$$
$$= \frac{8AC}{\varepsilon} \cdot \max\left\{\left\lceil \frac{L}{\alpha(\varepsilon)} \right\rceil, \frac{a}{\alpha(\varepsilon)^b}\right\}$$

Substituting in the definition of Ω_1 together with the fact that $\varepsilon \geq e^{-1/\varepsilon}$ we have

$$\alpha(\varepsilon) = \frac{\varepsilon}{4B^{8C/\varepsilon}} \geq K_1^{-1/\varepsilon}$$

for a suitable constant K_1 dependent on B and C, and therefore:

$$\psi(\varepsilon) \leq \frac{8AC}{\varepsilon} \cdot \max\{\lceil LK_1^{1/\varepsilon} \rceil, aK_1^{b/\varepsilon}\}$$

which means that we can find a sufficiently large constant K in terms of A, B, C, L, a and b such that

$$\psi(\varepsilon) \leq K^{1/\varepsilon}$$

and so we're done. □

We now show that this latter result is simply a reformulation, using 'proof-theoretic' rates of convergence, of the standard remainder estimate for Littlewood's theorem:

Proof of Theorem 2.4 from Corollary 6.4. Suppose that

$$F(x) = s + \mathcal{O}((1-x)^b)$$

for some $b > 0$, or equivalently

$$|F(x) - s| \leq a(1-x)^b$$

for some $a > 0$. Let $M \in \mathbb{N}$ be arbitrary and suppose that $x, y \in [e^{-1/M}, 1)$. Then

$$|F(x) - F(y)| \leq |F(x) - s| + |s - F(y)|$$
$$\leq 2a(1 - e^{-1/M})^b \leq \frac{2a}{M^b}$$

for the last line using the inequality $1 + x \leq e^x$ for $x = -1/M$. But then this implies that $|F(x) - F(y)| \leq \delta$ for any $x, y \in [e^{-1/M}, 1)$, provided that

$$M \geq (2a/\delta)^{1/b}$$

Therefore $\phi(\delta) = (2a)^{1/b}\delta^{-1/b}$ is a rate of convergence for $F(x)$ as $x \to 1^-$ in our sense. By Corollary 6.4 it follows that there is some constant $K > 0$ such that for any $\varepsilon \in (0,1]$ we have

$$\forall m, n \geq K^{2/\varepsilon}(|s_n - F(e^{-1/m})| \leq \varepsilon/2)$$

Choosing $m \in \mathbb{N}$ so that $|F(e^{-1/m}) - s| \leq \varepsilon/2$, we therefore have

$$\forall n \geq K^{2/\varepsilon}(|s_n - s| \leq \varepsilon)$$

for any $\varepsilon \in (0,1]$. We can invert this, since fixing $n \in \mathbb{N}$ and setting $\varepsilon := 2\log(K)/\log(n)$ given us $K^{2/\varepsilon} = n$, and thus for sufficiently large n we have

$$|s_n - s| \leq \frac{2\log(K)}{\log(n)}$$

and from this it follows that

$$s_n = s + \mathcal{O}\left(\frac{1}{\log(n)}\right)$$

and so Theorem 2.4 is proved. □

6.2 A remainder theorem for rates of metastability

Our first remainder theorem applies to the situation where both $F(x)$ and s_n converge with computable rates. However, it is well known that it is not in general the case that convergent sequences possess computable rates of convergence. The canonical counterexamples here are so-called Specker sequences [24], bounded and monotonically increasing sequences of rational numbers whose limit is not a computable real number. This phenomenon can be lifted to the case of power series as follows:

Proposition 6.5. *There exists a sequence $\{a_n\}$ with $a_n = \mathcal{O}(1/n)$, such that $F(x)$ converges as $x \to 1^-$, but with no computable rate of convergence in the sense of Definition 6.2.*

Proof. We make use of the proof of Proposition 3.2 of [22]. There it was shown that for any Specker sequence $\{q_n\}$, defining

$$a_n := \frac{q_{m+1} - q_m}{2^{m-1}} \quad \text{for } m = \lceil \log_2(n) \rceil$$

we obtain a sequence satisfying $a_n = o(1/n)$ (and so in particular $a_n = \mathcal{O}(1/n)$) and $s_{2n} = q_{n+1}$, and so therefore $s_n \to q$ where q is the noncomputable limit of the Specker sequence. This then implies that s_n has no computable rate of convergence. Now, by Abel's theorem (cf. Theorem 2.1), since s_n converges then so does $F(x)$ as $x \to 1^-$. But supposing for that $F(x)$ possesses a computable rate of convergence. Then by Theorem 6.3, $\{s_n\}$ also has a computable rate of convergence, a contradiction. □

The above proposition shows us that there are situations where Theorem 6.3 cannot apply, where $F(x) \to s$ as $x \to 1^-$ but with no computable rate of convergence. However, we now show that we can use our finitary Tauberian theorem also in the case where we do not have computable rates, to instead convert computable rates of *metastability* for $F(x) \to s$ to corresponding rates of metastability for $s_n \to s$. First of all, we need to define what a rate of metastability means in this context. Cauchy convergence in general is a $\forall\exists\forall$ statement i.e. of the form

$$\forall\varepsilon\exists N\forall k\, P(\varepsilon, N, k) \tag{24}$$

for some computationally neutral inner formula P. The existence of Specker sequences demonstrates that there are instances of formulas of this logical form where we cannot produce a computable bound on witnesses satisfying its Dialectica interpretation i.e. there is no computable ϕ satisfying

$$\forall\varepsilon\exists N \leq \phi(\varepsilon)\forall k\, P(\varepsilon, N, k)$$

Generally, this is because the proofs of such statements use classical reasoning, and so the intuitionistic soundness theorem which would normally imply the existence of a computable ϕ does not apply. However, we can instead apply the Dialectica interpretation in its *classical* form, by precomposing formulas with a negative translation. In particular, it is typically the case that for $\forall\exists\forall$ formulas, the following is provable intuitionistically, even when (24) is not:

$$\forall\varepsilon\neg\neg\exists N\forall k\, P(\varepsilon, N, k) \tag{25}$$

Applying the Dialectica interpretation to (25), we are instead asked to find a witness for N in the following formula:

$$\forall\varepsilon, g\exists N\, P(\varepsilon, N, g(N))$$

In the case of Cauchy convergence, a functional $\Phi : (0, \infty) \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ that bounds N in terms of ε and g i.e.

$$\forall\varepsilon, g\exists N \leq \Phi(\varepsilon, g)\, P(\varepsilon, N, g(N)) \tag{26}$$

is known as a *rate of metastability*. Metastable convergence theorems and the key role that they play in analysis as finitizations of 'soft' convergence statements is discussed in an essay by T. Tao [25] and used in [26], and the connection with the Dialectica interpretation is explored in [10] and particularly [5]. It is usually possible to extract rates of metastability from convergence proofs even when direct rates of convergence are not possible, and the extraction of such metastable bounds is a standard result in applied proof theory [11, 12, 21]. We can define rates of metastability for the two relevant convergence properties here following the pattern described above, and these should both be viewed as bounds on witnessing terms for the combined negative translation plus Dialectica interpretation of the respective properties.

Definition 6.6. We define a rate of metastability for $F(x)$ as $x \to 1^-$ to be any functional $\Phi : (0, \infty) \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ satisfying

$$\begin{aligned}&\forall\delta > 0, h : \mathbb{N} \to \mathbb{N}\exists M \leq \Phi(\delta, h)\\&\forall x, y \in [e^{-1/M}, e^{-1/(M+h(M))}](|F(x) - F(y)| \leq \delta)\end{aligned}$$

Similarly, a rate of metastability for $s_n \to \lim_{x\to 1^-} F(x)$ as $n \to \infty$ is defined to be any function $\Psi : (0,\infty) \times (\mathbb{N} \to \mathbb{N}) \to \mathbb{N}$ satisfying

$$\forall \varepsilon > 0, g : \mathbb{N} \to \mathbb{N} \exists N \leq \Psi(\varepsilon, g)$$
$$\forall m, n \in [N, N + g(N)](|s_n - F(e^{-1/m})| \leq \varepsilon)$$

We say that a rate of metastability is *computable* if, when restricted to rational inputs $\varepsilon \in (0,\infty)$, it forms a computable functional relative to the oracle $g : \mathbb{N} \to \mathbb{N}$ in the usual sense.

We now generalise our first remainder theorem so that it also applies in the case where $F(x) \to s$ as $x \to 1^-$, without a computable rate of convergence but with a computable rate of metastability.

Theorem 6.7 (Second remainder theorem)**.** *Suppose that there exists some $C > 0$ such that $n|a_n| \leq C$ for all $n \in \mathbb{N}$, and let $L > 0$ be a bound on $|F(x)|$ for $x \in [0,1)$. Suppose that Φ is a computable rate of metastability for $F(x)$ as $x \to 1^-$. Then a computable rate of metastability for $s_n \to \lim_{x\to 1^-} F(x)$ is given by*

$$\Psi(\varepsilon, g) := \beta(\varepsilon, \Phi(\alpha(\varepsilon), h_{\varepsilon,g}))$$

for $h_{\varepsilon,g} : \mathbb{N} \to \mathbb{N}$ defined by

$$h_{\varepsilon,g}(k) := \gamma(\alpha(\varepsilon), k, g(\beta(\varepsilon, k)))$$

where α, β and γ are defined as in Corollary 6.1.

Proof. Again, letting $B(\delta, M, l)$ and $D(\varepsilon, N, l)$ be defined as in Section 5.1, if Φ is a rate of metastability for $F(x)$ as $x \to 1^-$ then

$$\forall \delta > 0, h \exists M \leq \Phi(\delta, h)\, B(\delta, M, h(M))$$

In particular, for any $\varepsilon > 0$ and $g : \mathbb{N} \to \mathbb{N}$, setting $\delta = \alpha(\varepsilon)$ and $h = h_{\varepsilon,g}$ there exists $M \leq \Phi(\alpha(\varepsilon), h_{\varepsilon,g})$ such that

$$B(\alpha(\varepsilon), M, \gamma(\alpha(\varepsilon), M, g(\beta(\varepsilon, M))))$$

and therefore setting $N := \beta(\varepsilon, M)$, it follows from (23) that

$$D(\varepsilon, N, g(N))$$

By monotonicity of β we have

$$N = \beta(\varepsilon, M) \leq \beta(\varepsilon, \Phi(\alpha(\varepsilon), h_{\varepsilon,g})) = \Psi(\varepsilon, g)$$

and since ε and g were arbitrary we have shown that Ψ is a rate of metastability for $s_n \to \lim_{x\to 1^-} F(x)$. □

Remark 6.8. We observe that aside from generalising the traditional remainder estimate, Theorem 6.7 also gives us concrete witnesses to the Dialectica interpretation of Littlewood's theorem that could be applied as part of the extraction of witnessing terms from proofs where Littlewood's theorem is used in rule-form as a step in some nonconstructive argument.

7 Concluding remarks

In this paper, we have given a computational interpretation via Gödel's Dialectica interpretation to Littlewood's celebrated Tauberian theorem. The immediate relevance of this computational Tauberian theorem is demonstrated by rederiving standard remainder estimates and generalising them for arbitrary rates of convergence, and even the case where no computable rate of convergence exists. But more generally, we obtain a natural finitization of Littlewood's theorem, along with an intuitive constructive reading in terms of a winning strategy in a two player game, and we consider this to be of independent interest. It is also hoped that our case study is self-contained enough that it will form an useful illustration of how the Dialectica interpretation can be applied in mathematics to obtain quantitative information from proofs.

However, above all we see this paper as a forming a stepping stone to deeper results in quantitative Tauberian theory, bringing initial ideas sketched in [22] to bear on a much more complex Tauberian theorem, and demonstrating in turn that the Dialectica can be used to both rederive and generalise known numerical results in this area. We propose Tauberian theory as a area where there is a great deal of potential for applying proof-theoretic methods, and conclude with the following questions:

1. Can we extend the ideas presented here to the more complex Tauberian theorems later proved by Hardy and Littlewood in e.g. [7], to integral analogues of Tauberian theorems using Karamata's method as discussed in [14, Chapter I.13–14], or to even deeper results in Tauberian theory involving Fourier transformations and Wiener kernels cf. [14, Chapter II]?

2. Can we use techniques from proof theory to make further contributions to Tauberian remainder theory? In particular, are there Tauberian theorems with no known remainder estimates, for which the application of proof-theoretic methods could produce not just generalisations of existing remainder estimates as in this case, but improved or even brand new remainder theorems?

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