Dependent choice as a termination principle

Thomas Powell

Abstract

We introduce a new formulation of the axiom of dependent choice, which can be viewed as an abstract termination principle that in particular generalises recursive path orderings, the latter being fundamental tools used to establish termination of rewrite systems. We consider several variants of our termination principle, and relate them to general termination theorems in the literature.

Keywords. Dependent choice Arithmetic in all finite types Termination Path orderings Higher order reverse mathematics

1 Introduction

Path orderings are a technique for proving that programs terminate. They include the well known multiset [5] and Knuth-Bendix [9] orderings, and are central to the theory of term rewriting, where they form reduction orderings that can be used to prove that rewrite systems terminate [6].

In order to establish that path orders are wellfounded, one typically appeals to the axiom of dependent choice in some form. Traditionally, this is via Kruskal's theorem and a clever combinatorial idea of Nash-Williams known as the minimal bad sequence argument [7], although the latter can be given a constructive flavour by using bar induction instead [8].

However, results in the other direction - by which we mean a proof of dependent choice using some termination principle - are rare. Dependent choice is often used as a convenient mathematical tool without considering whether or not wellfoundedness could be established in a weaker theory. Indeed, as shown by Buchholz [3], the termination of fixed term rewrite systems via recursive path orderings can in fact even be proven in a weak fragment of Peano arithmetic.

In this note, we single out a general termination principle TP phrased in a higher-order setting, and prove that it is instance-wise equivalent to dependent choice over Peano arithmetic in all finite types. We go on to explore several variants of this principle.

There are several motivating factors behind this work. Firsly, the fact that dependent choice can be viewed as a general wellfoundedness principle akin to recursive path orders is of theoretical interest in its own right, and by making this precise we introduce variants of dependent choice which have deep links to program termination. Our paper is close in spirit to [11], which establishes a two-way connection between a strong logical principle on one hand and a termination argument on the other, and in this sense our main results can be viewed as a contribution to reverse mathematics in a higher type setting [10].

Secondly, by looking at termination on a high level we are able to clarify the relationship between various wellfoundedness proofs for path orderings one finds in the literature. One variant of our termination principle in particular is based on the notion of a simplification order, and allows us to prove the abstract theorem of Goubault-Larrecq [8] as a direct corollary, from which wellfoundedness of many well known path orderings follows. We explain how several standard proofs of the latter can be seen as instances of general proof patterns on the abstract level.

Finally, because we present our termination principle as a formal extension of Peano arithmetic in all finite types, a theory which can be given a direct computational interpretation in Gödel's system T via the functional interpretation, we take a step towards connecting path orderings with higher-order recursion, which is something we briefly mention in Section 7. This would extend work begun in [12] and continued in [15], where derivation trees of finitely branching path orders are encoded via terms of system T.

2 Preliminaries

We begin by giving a brief overview of recursive path orderings. This is not strictly necessary for the results that follows, and is there just to provide some idea of the concrete termination principles we have in mind while working in our abstract setting. In Section 2.2 we define the main formal systems we will work in.

2.1 Recursive path orders

Let T denote the set of first-order terms build from some finite set of function symbols F and countable set of variables X. To each function symbol f with arity n, we assign a *lifting*, which is an operation that takes any binary relation $\succ$ on some $A \subseteq T$ and extends it to a relation $\succ_f$ on A^n, satisfying the property that if $\succ$ is wellfounded on A then $\succ_f$ is wellfounded on A^n. The recursive path order $\succ_{\rm rpo}$ on T with respect to some set of liftings is defined inductively over the structure of terms as follows: $t = f(t_1, \ldots, t_n) \succ_{\rm rpo} s$ if either

(i) $t_i \succeq_{\rm rpo} s$ for some $i = 1, \ldots, n$,

(ii) $s = g(s_1, \ldots, s_m)$ for some $f \succ_F g$, and $t \succ_{\rm rpo} s_i$ for all $i = 1, \ldots, m$,

(iii) $s = f(s_1, \ldots, s_n)$, $t \succ_{\rm rpo} s_i$ for all $i = 1, \ldots, n$, and $(t_1, \ldots, t_n) \succ_{{\rm rpo},f} (s_1, \ldots, s_n)$.

In the case where $\succ_f$ is the multiset resp. lexicographic extension of $\succ$ for all function symbols f, we obtain (variants of) the well-known multiset resp. lexicographic path orderings. The following results are standard in the theory of term rewriting (see [7] for example, or the standard text [1]).

Theorem 2.1. *The recursive path order is closed under substitution and contexts, and is well-founded.*

Corollary 2.2. *Let $\mathcal{R}$ be a finite term rewrite system such that whenever $l \to r$ is a rule of $\mathcal{R}$ then $l \succ_{\mathrm{rpo}} r$. Then the rewrite relation generated by $\mathcal{R}$ is well-founded.*

Recursive path orders allow us to verify that programs defined by a set of rewrite rules are terminating. For example, implementations of many basic primitive recursive functions can be dealt with by the multiset path ordering, while multiply recursive functions such as the Ackermann function are typically reducing under the lexicographic path ordering. The key feature of path orderings of this kind is that they allow us to prove that *recursively* defined programs terminate. This is the role played by clause (iii) above.

Very informally, the reason that $\succ_{\mathrm{rpo}}$ itself is wellfounded relies on the fact that whenever we have a sequence of recursive calls

$$f(t_1, \ldots, t_n) \succ_{\mathrm{rpo}} f(s_1, \ldots, s_n) \succ_{\mathrm{rpo}} \ldots$$

where each subterm in the sequence is well-founded with respect to $\succ_{\mathrm{rpo}}$, then that sequence must be finite since $\succ_{\mathrm{rpo},f}$ is a lifting. Such sequences are an example of what we will call *minimal* sequences.

This notion that wellfoundedness of an ordering is implied by wellfoundedness of its minimal sequences constitutes an idea far more general than the world of path orderings, and the purpose of this paper is to explore it on a much more abstract level. For more details on how our abstract results can be related to the more concrete material outlined in this section, the reader is encouraged to consult [8, Section 3], which shows how wellfoundedness of standard path orders follows as a direct consequence of Goubault-Larrecq's termination theorem, which is in turn derivable from our termination principles (cf. Corollary 6.1).

2.2 Extensions of Peano arithmetic in all finite types

The finite types are defined inductively as follows: `Nat` and `Bool` are types, and if ρ and τ are types then so is the function space $\rho \to \tau$ (which we sometimes write as τ^ρ), the cartesian product $\rho \times \tau$ and set of finite sequences ρ^*. The basic logical system we work in is the theory PA^ω of Peano arithmetic in all finite types, which is just the usual first order theory of Peano arithmetic but now with variables and quantifiers for all types. It also includes the usual combinators for the lambda calculus together with constants for primitive recursion in all types. For a more detailed outline of the kind of theory we have in mind see e.g. [19] - the precise setup is not important here. We leave open whether or not the equivalences that follow can be established in a strictly weaker higher order setting, such as Kohlenbach's RCA_0^ω [10]. For the purposes of this paper, PA^ω forms a convenient base theory over which we can reason about much stronger choice principles. We make use of the following notation: For $\alpha, \beta : \texttt{Nat} \to \rho$, $a, b : \rho^*$ and $x : \rho$

- $[\alpha](n) := [\alpha_0, \ldots, \alpha_{n-1}]$ denotes the initial segment of α of length n. The empty sequence (for any type) is denoted $[]$,
- $|a| : \texttt{Nat}$ denotes the length of a,
- $a * x := [a_0, \ldots, a_{k-1}, x]$ is the one element extension of a with x, and similarly $a * b$ resp. $a * \beta$ is the extension of a with the finite sequence b resp. infinite sequence β.
- $a \blacktriangleleft \alpha$ denotes that α is an extension of a i.e. $(\forall i < |a|)(a_i = \alpha_i)$,
- similarly, $a \blacktriangleleft b$ denotes that a is a (not necessarily strict) prefix of b i.e. $|a| \leq |b| \wedge (\forall i < |a|)(a_i = b_i)$.

The axiom schema of dependent choice of type ρ is given by

$$\mathrm{DC}_\rho \ : \ \forall n^{\texttt{Nat}} \, \forall x^\rho \, \exists y^\rho A(n, x, y) \to \exists \alpha^{\texttt{Nat} \to \rho} \forall n \, A(n, \alpha(n), \alpha(n+1))$$

where A is some formula in the language of PA^ω. Closely related to dependent choice is the principle of bar induction, which in this paper will be given as the following schema of relativised bar induction in all finite types:

$$\mathrm{RBI}_\rho \ : \ \left. \begin{cases} S([]) \\ \wedge \, (\forall \alpha^{\texttt{Nat} \to \rho} \in S)(\exists n) P([\alpha](n)) \\ \wedge \, (\forall a^{\rho^*} \in S)((\forall x^\rho)(S(a * x) \to P(a * x)) \to P(a)) \end{cases} \right\} \to P([]),$$

where P and S are formulas in the language of PA^ω, $a \in S$ is shorthand for $S(a)$ and $\alpha \in S$ shorthand for $(\forall n^0) S([\alpha](n))$. We denote by $\mathrm{PA}^\omega + \mathrm{DC}$ the extension of PA^ω with the axiom schemata DC_ρ for all finite types, and similarly for $\mathrm{PA}^\omega + \mathrm{RBI}$.

Let $\rhd : \rho \times \rho \to \texttt{Bool}$ be a binary relation on ρ, and let

$$\mathrm{TI}_\rho[\rhd] \ : \ \forall x^\rho (\forall y (x \rhd y \to A(y)) \to A(x)) \to \forall x A(x)$$

denote the usual principle of transfinite induction over $\rhd$. We denote by $\rhd_{\mathrm{lex}}$ the lexicographic extension of $\rhd$ to infinite sequences of type $\texttt{Nat} \to \rho$ i.e.

$$\alpha \rhd_{\mathrm{lex}} \beta :\equiv \exists n^{\texttt{Nat}} ([\alpha](n) = [\beta](n) \wedge \alpha(n) \rhd \beta(n)).$$

Open induction over $\rhd_{\mathrm{lex}}$ in all finite types [2] is given by the schema

$$\mathrm{OI}_\rho[\rhd] \ : \ \forall \alpha^{\rho^{\texttt{Nat}}} (\forall \beta (\alpha \rhd_{\mathrm{lex}} \beta \to U(\beta)) \to U(\alpha)) \to \forall \alpha U(\alpha)$$

where now $U(\alpha)$ is a so-called *open predicate*, which in a classical setting is defined to be one of the form $\exists n^{\texttt{Nat}} B([\alpha](n))$ for some arbitrary formula $B(s)$ on ρ^*.

Definition 2.3. For a pair of axiom schemas Σ_ρ and Δ_ρ, we say that $\Sigma_{\tau_\rho} \to \Delta_\rho$ instance-wise over some theory T if for any type ρ and instance Φ of the axiom schema Δ_ρ, there is an instance Ψ of the axiom schema Σ_{τ_ρ} such that $T \vdash \Psi \to \Phi$.

Theorem 2.4. *The following are provable over* PA^ω *and hold instance-wise:*

- $\mathrm{RBI}_\rho \leftrightarrow \mathrm{DC}_\rho$,
- $\mathrm{OI}_{\rho\times\mathtt{Bool}}[\rhd] \to \mathrm{DC}_\rho$, *where* $\rhd$ *is the relation on* $\rho\times\mathtt{Bool}$ *defined by* $(x,b)\rhd(x',b')$ *iff* $b=1$ *and* $b'=0$.

Over $\mathrm{PA}^\omega + \mathrm{TI}_\rho[\rhd]$ *we also have*

- $\mathrm{DC}_\rho \to \mathrm{OI}_\rho[\rhd]$.

Proof. That dependent choice proves bar induction is well-known. The remaining results follow from [2, Propositions 3.3-3.4]. □ □

3 The first termination principle TP

Let us now forget about terms and recursive path orderings, which can be encoded in PA^ω as objects and relations of type Nat, and replace these with some arbitrary type ρ and *binary predicate* $\succ$ on $\rho\times\rho$. Note that by this we mean that $x\succ y \leftrightarrow A(x,y)$ for some formula in the language of PA^ω, which is strictly more general than being a relation of the form $f(x,y)=0$: In particular, $x\succ y$ need not be decidable, though in the case of path orderings it is of course instantiated by a decidable predicate. We will in addition consider a binary relation $\rhd$ on ρ, which for path orderings plays the role of the subterm relation, although in our abstract setting is also entirely arbitrary.

Remark 3.1. Path orders $\succ$ are normally transitive, although here we do not assume this, since it plays no role in *termination.*

Definition 3.2. We say that the sequence $\alpha:\mathtt{Nat}\to\rho$ is wellfounded (w.r.t. $\succ$) if it satisfies the predicate $\mathit{WF}_\rho[\succ](\alpha)$ defined by

$$\mathit{WF}_\rho[\succ](\alpha) :\equiv \exists n(\alpha_n \not\succ \alpha_{n+1}).$$

The relation $\succ$ is wellfounded if $\forall\alpha\,\mathit{WF}_\rho[\succ](\alpha)$.

Definition 3.3. We say that an infinite sequence α is *minimal* with respect to $\rhd$ if all sequences lexicographically less than α are wellfounded with respect to $\succ$. We write this formally via the predicate $\mathit{MIN}_\rho[\rhd,\succ]$ given by

$$\mathit{MIN}_\rho[\rhd,\succ](\alpha) :\equiv \forall\beta(\alpha \rhd_{\mathrm{lex}} \beta \to \mathit{WF}_\rho[\succ](\beta)).$$

Our first abstract termination principle is nothing more than a formalisation of the idea briefly discussed in Section 2.1, namely the statement that if all minimal sequences are well-founded, then $\succ$ is well-founded.

Definition 3.4 (Termination principle). Given a relation $\rhd$ on ρ, we define the schema $\mathrm{TP}_\rho[\rhd]$ as follows:

$$\mathrm{TP}_\rho[\rhd] :\equiv \forall\alpha(\mathit{MIN}_\rho[\rhd,\succ](\alpha) \to \mathit{WF}_\rho[\succ](\alpha)) \to \forall\alpha\,\mathit{WF}_\rho[\succ](\alpha)$$

where $x^\rho \succ y^\rho$ is an arbitrary formula in the language of PA^ω.

In the remainder of this paper we will drop the subscripts and/or parameters on TP, $MIN(\alpha)$ and $WF(\alpha)$ whenever there is no risk of ambiguity.

We now show that TP has the same strength as the axiom of dependent choice and its variants, and is, moreover, instance-wise equivalent to each of the three choice principles outlined in Section 2.2. This will follow from Theorem 2.4, together with the observation that TP_ρ is instance-wise equivalent to OI_ρ, which we now prove.

Lemma 3.5. $\mathrm{OI}_\rho[\rhd] \to \mathrm{TP}_\rho[\rhd]$ *instance-wise over* PA^ω.

Proof. This is clear, since $WF[\succ](\alpha)$ can be expressed as the open predicate

$$\exists n B([\alpha](n)) \quad \text{for} \quad B(s) :\equiv s_{|s|-2} \not\succ s_{|s|-1}$$

and $\mathrm{TP}[\rhd]$ is then nothing more than $\mathrm{OI}[\rhd]$ on the predicate $WF[\succ]$. □ □

Corollary 3.6. $\mathrm{DC}_\rho \to \mathrm{TP}_\rho[\rhd]$ *and* $\mathrm{RBI}_\rho \to \mathrm{TP}_\rho[\rhd]$ *instance-wise over* $\mathrm{PA}^\omega + \mathrm{TI}_\rho[\rhd]$.

Proof. Direct from Theorem 2.4 together with the above lemma. □ □

Note that the standard proofs of open induction from either dependent choice (via the minimal bad-sequence argument) or bar induction (see e.g. [2, 4]), both lift easily via Lemma 3.5 to the termination principle. We provide versions of these proofs explicitly in Appendix A, because they form an abstract instance of important patterns often encountered in the term-rewriting literature, as we will discuss in Section 6.

Lemma 3.7. $\mathrm{TP}_{\rho^*}[\rhd^*] \to \mathrm{OI}_\rho[\rhd]$ *for suitable* $\rhd^*$*, instance-wise over* PA^ω.

Proof. To derive $\mathrm{OI}[\rhd]$ on the open predicate $U(\alpha) :\equiv \exists n^{\mathtt{Nat}} B([\alpha](n))$, define $\succ$ and $\rhd^*$ on ρ^* by

$$\begin{aligned} a \succ b &:\equiv (|b| = |a| + 1) \wedge (a \blacktriangleleft b) \wedge (\forall c \blacktriangleleft b)\neg B(c) \\ a \rhd^* b &:\equiv (|b| \geq |a|) \wedge (\exists i < |a|)([a](i) = [b](i) \wedge a_i \rhd b_i). \end{aligned}$$

where we recall that $\blacktriangleleft$ is the prefix relation on sequences. Assuming the premise of $\mathrm{OI}[\rhd]$ we prove the premise of $\mathrm{TP}[\rhd^*]$. Take some minimal $\gamma \in \mathtt{Nat} \to \rho^*$, which then satisfies

$$(\forall \delta)(\gamma \rhd^*_{\mathrm{lex}} \delta \to \exists n(\delta_n \not\succ \delta_{n+1})). \tag{1}$$

We want to prove $\exists n(\gamma_n \not\succ \gamma_{n+1})$.

To this end, we can assume w.l.o.g. that $\forall n(|\gamma_{n+1}| = |\gamma_n| + 1 \wedge \gamma_n \blacktriangleleft \gamma_{n+1})$, else if this were false then by definition there would be some n with $\gamma_n \not\succ \gamma_{n+1}$. Let $N := |\gamma_0|$ and define the diagonal sequence $\tilde{\gamma} \in \rho^{\mathtt{Nat}}$ by

$$\tilde{\gamma}_n := \begin{cases} (\gamma_0)_n & \text{if } n < N \\ (\gamma_{m+1})_{N+m} & \text{if } n = N + m, \end{cases}$$

which is well-defined since $|\gamma_m| = N + m$.

Now suppose that $\beta \in \rho^{\mathtt{Nat}}$ is such that $\tilde{\gamma} \rhd_{\mathrm{lex}} \beta$, and define $\delta : (\rho^*)^{\mathtt{Nat}}$ by $\delta_n := [\beta](N+n)$ (so in particular $|\gamma_n| = N + n = |\delta_n|$ for all n). Then we must have $\gamma \rhd^*_{\mathrm{lex}} \delta$.

To see this, recall that $\tilde{\gamma} \rhd_{\mathrm{lex}} \beta$ means there exists some m with $[\tilde{\gamma}](m) = [\beta](m)$ and $\tilde{\gamma}_m \rhd \beta_m$. Then either $m < N$, in which case we have $[\gamma_0](m) = [\tilde{\gamma}](m) = [\beta](m) = [\delta_0](m)$ and $(\gamma_0)_m \rhd (\delta_0)_m$ and so $\gamma_0 \rhd^* \delta_0$, or $m = N + k$ for some k, and since $[\tilde{\gamma}](N+k) = [\beta](N+k)$ and $\tilde{\gamma}_{N+k} \rhd \beta_{N+k}$ it follows that $\gamma_n = \delta_n$ for all $n \leq k$, and in addition $[\gamma_{k+1}](N+k) = [\delta_{k+1}](N+k)$ and $(\gamma_{k+1})_{N+k} \rhd (\delta_{k+1})_{N+k}$ and hence $\gamma_{k+1} \rhd^* \delta_{k+1}$, which all together implies $\gamma \rhd^*_{\mathrm{lex}} \delta$.

But if $\gamma \rhd^*_{\mathrm{lex}} \delta$ then by (1) there is some n with $\delta_n \not\succ \delta_{n+1}$, and since $\delta_n = [\beta](N+n) \blacktriangleleft [\beta](N+n+1) = \delta_{n+1}$ this means that $(\exists c \blacktriangleleft [\beta](N+n+1))B(c)$, or in other words, $B([\beta](k))$ must hold for some $k \leq N + n + 1$, from which $U(\beta)$ follows by definition.

Therefore we have shown that $\forall \beta(\tilde{\gamma} \rhd_{\mathrm{lex}} \beta \to U(\beta))$, and hence by the premise of $\mathrm{OI}[\rhd]$ we obtain $U(\tilde{\gamma})$. But this means that there is some n such that $B([\tilde{\gamma}](n))$ holds, and since

$$[\tilde{\gamma}](n) \blacktriangleleft \gamma_{n \dot{-} N} \blacktriangleleft \gamma_{(n \dot{-} N)+1}$$

(where $\dot{-}$ denotes truncated subtraction) it follows that $(\exists c \blacktriangleleft \gamma_{(n \dot{-} N)+1})B(c)$, and therefore must have $\gamma_{n \dot{-} N} \not\succ \gamma_{(n \dot{-} N)+1}$. Therefore we have shown in all cases that $\exists n(\gamma_n \not\succ \gamma_{n+1})$ whenever γ is minimal with respect to $\rhd^*$. This establishes the premise of $\mathrm{TP}[\rhd^*]$, and so it follows that $(\exists n)(\gamma_n \not\succ \gamma_{n+1})$ holds for *arbitrary* γ.

Now take some $\alpha \in \rho^{\mathtt{Nat}}$ and define γ by $\gamma_n = [\alpha](n)$. Since there exists some n with $\gamma_n \not\succ \gamma_{n+1}$ it follows that $(\exists c \blacktriangleleft [\alpha](n+1))B(c)$, in other words $B([\alpha](k))$ holds for $k \leq n+1$, and thus $U(\alpha)$ holds. Therefore using $\mathrm{TP}[\rhd^*]$ we have proved $\mathrm{OI}_\rho[\rhd]$. □ □

A direct result of Lemma 3.7 together with Theorem 2.4 is the following:

Corollary 3.8. $\mathrm{TP}_{(\rho \times \mathtt{Bool})^*}[\rhd^*] \to \mathrm{DC}_\rho$ *instance-wise over* PA^ω *where* $\rhd^*$ *is defined as in Lemma 3.7 from* $\rhd$ *on* $\rho \times \mathtt{Bool}$ *as in Theorem 2.4.*

4 An equivalent formulation of TP for well-founded elements

So far, our termination principle is essentially a modification of open induction, for open predicates which are restricted to two consecutive elements. In this section, we reformulate TP so that it more closely resembles a genuine termination argument, in the sense that it deals with well-founded *elements* of ρ rather than sequences. We will then apply this in the next section to provide an abstract termination principle for a generalisation of simplification orders.

Definition 4.1. We say that $x : \rho$ is well-founded if it satisfies the predicate

$$WF^*_\rho[\succ](x) :\equiv \forall \alpha(x \blacktriangleleft \alpha \to WF_\rho[\succ](\alpha)),$$

where here we use the strict prefix notation $x \blacktriangleleft \alpha$ to denote $x = \alpha_0$.

Definition 4.2. We define

$$MIN^*_\rho[\rhd, \succ](\alpha) :\equiv \forall n, y^\rho(\alpha_{n-1} \succ y \wedge \alpha_n \rhd y \to WF^*_\rho[\succ](y)))$$

where for $n = 0$ the condition $\alpha_{n-1} \succ y$ vanishes.

Definition 4.3. The termination principle $\mathrm{TP}^*_\rho[\rhd]$ is defined as

$$\mathrm{TP}^*_\rho[\rhd] :\equiv \forall \alpha(MIN^*_\rho[\rhd, \succ](\alpha) \to WF_\rho[\succ](\alpha)) \to \forall x\, WF^*_\rho[\succ](x).$$

where $\succ$ is an arbitrary formula in the language of PA^ω.

Lemma 4.4. $\mathrm{TP}_\rho[\rhd] \leftrightarrow \mathrm{TP}^*_\rho[\rhd]$ *instance-wise over* PA^ω.

Proof. We clearly have $\forall \alpha\, WF(\alpha) \leftrightarrow \forall x\, WF^*(x)$ and so the result follows if we can show that the premise of TP is equivalent to that of TP^*.

In one direction, assume that $\forall \alpha(MIN(\alpha) \to WF(\alpha))$ and $MIN^*(\alpha)$ holds for some fixed α. Take some $\beta \lhd_{\mathrm{lex}} \alpha$ with $[\alpha](n) = [\beta](n) \wedge \alpha_n \rhd \beta_n$. We show that $WF(\beta)$. Either $\beta_{n-1} \not\succ \beta_n$ and we're done, or $\alpha_{n-1} = \beta_{n-1} \succ \beta_n$, and since in addition $\alpha_n \rhd \beta_n$, by $MIN^*(\alpha)$ we have $WF^*(\beta_n)$, and so in particular since $\beta_n \blacktriangleleft \gamma$ for $\gamma_k := \beta_{n+k}$ we have $\beta_{n+k} \not\succ \beta_{n+k+1}$ for some $k \in \mathtt{Nat}$ and hence $WF(\beta)$ (note that for $n = 0$, $WF(\beta)$ follows directly from $\alpha_0 \rhd \beta_0$, since in this case the requirement $\alpha_{n-1} \succ \beta_n$ vanishes). This establishes $MIN(\alpha)$ and therefore $WF(\alpha)$.

For the other direction, assume that $\forall \alpha(MIN^*(\alpha) \to WF(\alpha))$ and $MIN(\alpha)$ holds for some fixed α, and suppose for contradiction that $\neg WF(\alpha)$. Take some n and y such that $\alpha_{n-1} \succ y$ and $\alpha_n \rhd y$. Then in particular, for any β we have $[\alpha](n) * y * \beta \lhd_{\mathrm{lex}} \alpha$ and therefore $WF([\alpha](n) * y * \beta)$ by $MIN(\alpha)$. But since $\alpha_0 \succ \ldots \succ \alpha_{n-1} \succ y$, this means that $WF(y * \beta)$, and thus we have established $WF^*(y)$ and hence $MIN^*(\alpha)$. But this implies $WF(\alpha)$, a contradiction. □ □

5 Simplification orders

We now present our final variation of TP, which as we will show can be directly related to abstract termination principles as they appear in the term-rewriting literature. The key to this is to introduce an additional property in all finite types which generalises a feature possessed by the majority of well-known path orders in term rewriting: namely that $x \rhd y$ - or more generally $(\exists u)(x \rhd u \succeq y)$ - implies $x \succ y$. Recall that in this setting, $\rhd$ plays the role of the subterm relation, and orders which have the aforementioned property over a term structure are known as *simplification* orders.

Simplification orders can be characterised by a auxiliary relation $\succ_0$ which essentially defines $x \succ y$ in the case that $(\exists u)(x \rhd u \succeq y)$ is not true. In the case of terms in [7], this splitting up of $\succ$ is called a *decomposition*, and so we use the same terminology here, although of course for us our basic objects are not terms but elements of some arbitrary type ρ, and our relation $\succ$ can be a predicate of arbitrary logical complexity.

Definition 5.1. A predicate $\succ_0$ on $\rho \times \rho$ is called a decomposition of $\succ$ with respect to $\rhd$ if it satisfies the following two properties:

(a) $x \succ y \to \exists u(x \rhd u \succeq y) \vee x \succ_0 y$,

(b) $x \succ_0 y \to \forall u(y \rhd u \to x \succ u)$,

where $\succeq$ denotes the predicate $x \succ y \vee x = y$. Note that if $x \succ y \to \forall u(y \rhd u \to x \succ u)$ then $\succ$ is a decomposition of itself, although naturally we are interested in cases where $\succ_0$ is a not the same as $\succ$.

Example 5.2. For the recursive path order discussed in Section 2.1, we would define $t = f(t_1, \ldots, t_n) \succ_0 s$ iff

(i) $s = g(s_1, \ldots, s_m)$ for some $f \succ_F g$, and $t \succ_{\mathrm{rpo}} s_i$ for all $i = 1, \ldots, m$,

(ii) $s = f(s_1, \ldots, s_n)$, $t \succ_{\mathrm{rpo}} s_i$ for all $i = 1, \ldots, n$, and $(t_1, \ldots, t_n) \succ_{\mathrm{rpo},f} (s_1, \ldots, s_n)$.

Then $\succ_0$ is clearly a decomposition of $\succ_{\mathrm{rpo}}$ with respect to the immediate subterm relation $\rhd$.

The notion of a decomposition is extremely useful, as it enables us to restrict our attention to wellfoundedness of minimal sequences under the auxiliary relation $\succ_0$, which in practise is usually chosen to be something obviously wellfounded.

Definition 5.3. Define the predicate $A_\rho[\rhd, \succ](x)$ on ρ by

$$A_\rho[\rhd, \succ](x) :\equiv (\forall y \lhd x)\, WF^*_\rho[\succ](y)$$

and define

$$WF^*_{\rho,A}[\rhd, \succ, \succ_0](x) :\equiv (\forall \alpha \in A_\rho[\rhd, \succ])(x \blacktriangleleft \alpha \to WF_\rho[\succ_0](\alpha)).$$

where $\alpha \in A$ is shorthand for $\forall n A(\alpha_n)$.

Definition 5.4. The termination principle $\mathrm{TP}^s_\rho[\rhd]$ is defined as

$$\mathrm{TP}^s_\rho[\rhd] :\equiv \forall x\, WF^*_{\rho,A}[\rhd, \succ, \succ_0](x) \to \forall x\, WF^*_\rho[\succ](x)$$

where $\succ$ and $\succ_0$ range over arbitrary formulas in the language of PA^ω.

Theorem 5.5. *If $\succ_0$ is a decomposition of $\succ$ with respect to $\rhd$, then* $\mathrm{TP}^*[\rhd] \to \mathrm{TP}^s[\rhd]$ *instance-wise over* PA^ω. *If, in addition, $x \succ_0 y \to x \succ y$ then the implication holds in the other direction.*

Proof. For one direction suppose that TP^* and $\forall x\, WF^*_A(x)$ hold. We fix some α and prove $MIN^*(\alpha) \to WF(\alpha)$. Suppose for contradiction that $\neg WF(\alpha) \wedge MIN^*(\alpha)$ is true. Our first step is to show that $\neg WF[\succ_0](\alpha)$. Suppose for contradiction that $\alpha_n \not\succ_0 \alpha_{n+1}$ for some n, and w.l.o.g. take this n to be minimal. Then since we must have $\alpha_n \succ \alpha_{n+1}$ (by $\neg WF(\alpha)$), by property (a) it can only be that $\alpha_n \rhd u \succeq \alpha_{n+1}$ for some u. But by minimality of n we have $\alpha_{n-1} \succ_0 \alpha_n$ and hence by property (b) of Definition 5.1 we have $\alpha_{n-1} \succ u$. But since both $\alpha_{n-1} \succ u$ and $\alpha_n \rhd u$ it follows from $MIN^*(\alpha)$ that $WF^*(u)$, and since $u \succeq \alpha_{n+1}$ this implies that $WF^*(\alpha_{n+1})$ and hence $WF(\alpha)$ (note that in the case $n = 0$, the above argument simplifies since the prerequisite $\alpha_{n-1} \succ u$ is now redundant). But this contradicts our assumption $\neg WF(\alpha)$, and thus we can infer $\neg WF[\succ_0](\alpha)$.

Now, it follows from $MIN^*(\alpha)$ that for any n, y we have $\alpha_{n-1} \succ y \wedge \alpha_n \rhd y \to WF^*(y)$. But since by $\neg WF[\succ_0](\alpha)$ we must have $\alpha_{n-1} \succ_0 \alpha_n$, by property (b) it follows that $\alpha_n \rhd y$ automatically implies $\alpha_{n-1} \succ y$, and so in summary we have shown $(\forall n, y)(\alpha_n \rhd y \to WF^*(y))$, or in other words $\alpha \in A[\rhd, \succ]^{\mathbb{N}}$. But then $\neg WF[\succ_0](\alpha)$ contradicts $WF^*_A(\alpha_0)$ and thus also our assumption that $\forall x\, WF^*_A(x)$. So $\neg WF(\alpha) \wedge MIN^*(\alpha)$ must be false, and since α was arbitrary we have proven the premise of TP^*, from which we can infer $\forall x\, WF^*(x)$ and hence TP^s.

For the other direction, given our additional assumption $x \succ_0 y \to x \succ y$, suppose that TP^s and the premise of TP^* hold. Let's take some $x \blacktriangleleft \alpha$ with $\alpha \in A^{\mathbb{N}}$. Then it is clear that such an α must satisfy $MIN^*(\alpha)$: Given n and y with $\alpha_{n-1} \succ y$ and $\alpha_n \rhd y$, then by $\alpha_n \in A$ we clearly have $WF^*(y)$. Therefore by the premise of TP^* we have $\alpha_n \not\succ \alpha_{n+1}$ for some n, and by our additional assumption this implies $\alpha_n \not\succ_0 \alpha_{n+1}$ and hence $WF[\succ_0](\alpha)$. Since x and α were arbitrary we have proved $\forall x\, WF^*_A(x)$ from which we can infer $\forall x\, WF^*(x)$ by TP^s, and this establishes TP^*. □ □

6 A connection with abstract path orderings

Our final termination principle TP^s follows instance-wise from dependent choice and conversely, modulo a small additional assumption, the two are actually equivalent. We now show how TP^s can very much be viewed as a genuine termination principle by showing that it can be used to prove the abstract termination theorem of Goubault-Larrecq [8, Theorem 1], when the latter is formulated here in a typed setting and using our notational conventions (cf. Definition 5.3 in particular).

Corollary 6.1 (Goubault-Larrecq [8], typed variant)**.** *Let $\succ$, $\rhd$ and $\gg$ be three binary relations on ρ such that $x \succ y$ implies that either*

(i) $x \rhd u \succeq y$ for some u, or

(ii) $x \gg y$ and $(\forall u)(y \rhd u \to x \succ u)$.

Assume in addition that

(iii) $\mathrm{TI}_\rho[\rhd]$ *holds, and*

(iv) for every $x : \rho$ *we have* $WF^*_{\rho,A}[\rhd, \succ, \gg](x)$.

Then $\forall x\, WF^*[\succ](x)$ *over* $\mathrm{PA}^\omega + \mathrm{DC}_\rho$.

Remark 6.2. Our condition (iii) is formulated as wellfoundedness of $\rhd$ in [8], while our condition (iv) represents Goubault-Larrecq's alternative condition (v) (cf. [8, Remark 6]). There, $x \in SN$ corresponds to our predicate $WF^*[\succ](x)$, while the set $\overline{SN}$ corresponds to $\{x \in \rho \; : \; A_\rho[\rhd, \succ](x)\}$. Thus his condition reads that for any x satisfying $A[\rhd, \succ](x)$, whenever $x \blacktriangleleft \alpha$ for $\alpha \in A[\rhd, \succ]$ then $WF[\gg](\alpha)$, which following the notation of Definition 5.3 is just $WF^*_A[\rhd, \succ, \gg](x)$.

Proof. The first assumption that $x \succ y$ implies either (i) or (ii) shows that the binary predicate $\succ_0$ given by

$$x \succ_0 y :\equiv x \gg y \wedge (\forall u)(y \rhd u \to x \succ u)$$

is a decomposition of $\succ$ with respect to $\rhd$. Note that since $x \succ_0 y \to x \gg y$, then $WF[\gg](\alpha)$ implies $WF[\succ_0](\alpha)$ for any α. Since by condition (iv) we have that for any x and $\alpha \in A[\rhd, \succ]$ with $x \blacktriangleleft \alpha$ that $WF[\gg](\alpha)$, then also $WF[\succ_0](\alpha)$, from which we infer $\forall x\, WF^*_{\rho,A}[\rhd, \succ, \succ_0](x)$. Therefore $\forall x\, WF^*_\rho[\succ](x)$ follows directly from $\mathrm{TP}^s_\rho[\rhd]$. To show that the result is provable in $\mathrm{PA}^\omega + \mathrm{DC}_\rho$, we simply backtrack through the relevant implications set out earlier in the paper: It sufficies to establish $\mathrm{TP}^s_\rho[\rhd]$, which by Theorem 5.5 follows from $\mathrm{TP}^*_\rho[\rhd]$, using that $\succ_0$ is a decomposition of $\succ$ with respect to $\rhd$. But by Lemma 4.4 this in turn follows from $\mathrm{TP}_\rho[\rhd]$, which by Corollary 3.6 in turn is provable from DC_ρ over $\mathrm{PA}^\omega + \mathrm{TI}_\rho[\rhd]$, and since we take $\mathrm{TI}_\rho[\rhd]$ as an assumption, the result is provable in $\mathrm{PA}^\omega + \mathrm{DC}_\rho$. □ □

The original proof in [8] uses a variant of bar induction. If we were to take the bar inductive proof of TP (given explicitly in Appendix A) and adapt this via the proofs of Lemma 4.4 and Theorem 5.5 to a bar inductive proof of TP^s, it would be closely related to that of [8].

In addition, [8] shows that wellfoundedness of many of the usual path orders, including Fereirra-Zantema's wellfoundedness proof for term orderings [7, Theorem 4], follow as a corollary of the above result, and so in turn must also be subsumed by our abstract termination principle (even just restricted to $\rho := \mathtt{Nat}$). Moreover, were we to adapt the proof of TP via dependent choice (also given in Appendix A) to prove Theorem 4 of [7], we would end up with a very similar proof based on a minimal bad-sequence construction.

It is interesting to note that although proofs of termination via open induction are much less common, they have been considered from the perspective of formalisation [18], where direct inductive argument is much easier to work with than proof which is reliant on classical logic.

All of this demonstrates that TP and its variants are not only abstract termination theorems in the sense that they subsume well known termination results in the literature, but also the proofs of TP via DC and RBI can be seen as abstract representations of common proof techniques seen in the theory of term rewriting. We hope that, among other things, this note is helpful in making all of this precise.

To summarise, we have the following chain of termination principles, starting at the most general:

$$\begin{aligned} \mathrm{DC} \leftrightarrow \mathrm{TP} \leftrightarrow \mathrm{TP}^* \leftrightarrow \mathrm{TP}^s & \\ & \to \text{Goubault-Larrecq [8, Theorem 1]} \\ & \to \text{Fereirra-Zantema [7, Theorem 4]} \\ & \to \text{multiset, lexicographic path orders etc.} \end{aligned}$$

Note that from Fereirra-Zantema onwards, the termination theorems deal specifically with terms over some signature and primitive recursive relations, and can thus be encoded using just $\mathrm{TP}_{\mathtt{Nat}}$ of base type for decidable predicates $\succ$.

7 Concluding remarks

It is hoped that this short note provides some insight into proof theoretic aspects of termination arguments commonly found in term rewriting and related areas, in particular their relation to choice principles.

An interesting next step would be to consider forms of higher-order recursion which constitute natural computational counterparts to our termination principles. For the axiom of open induction, a corresponding recursor called *open recursion* has been considered by Berger [2] and shown to give a direct realizability interpretation to $\mathrm{OI}[\rhd]$.

It is easy to see that essentially the same form of recursion would give a computational interpretation to $\mathrm{TP}[\rhd]$: For a functional $f : \rho^{\mathtt{Nat}} \to (\mathtt{Nat} \to \rho \to \rho^{\mathtt{Nat}} \to \mathtt{Nat}) \to \mathtt{Nat}$ satisfying the modified realizability interpretation of the premise of TP, which is:

$$\begin{aligned} \forall \alpha, \phi (\forall n, y, \beta (\alpha_n \rhd y \to ([\alpha](n) * y * \beta)_{\phi n y \beta} \not\succ ([\alpha](n) * y * \beta)_{\phi n y \beta + 1}) \\ \to \alpha_{f\alpha\phi} \not\succ \alpha_{f\alpha\phi+1}) \end{aligned}$$

we would have $\forall \alpha (\alpha_{\Phi f \alpha} \not\succ \alpha_{\Phi f \alpha + 1})$, where here Φ denotes the functional recursively defined by

$$\Phi f \alpha = f \alpha (\lambda n, y, \beta \text{ . if } \alpha_n \rhd y \text{ then } \Phi f([\alpha](n) * y * \beta) \text{ else } 0).$$

Giving a computational interpretation to TP^* and TP^s, on the other hand, would be more difficult, since the proofs of these principles from TP seem to use classical logic in an essential way. One option would be to either use realizability together with the A-translation, or to consider instead the functional interpretation, which has been used to analyse the combination of open induction and classical logic in [16]

Particularly intriguing would be to see whether the equivalences proven here give rise to new interdefinability results between forms of recursion as in [14].

However, we leave such questions to future work.

Acknowledgements. I am grateful to the anonymous referees for their many comments and corrections. I am also indebted to Georg Moser for suggesting the proof theoretic study of abstract path orders, and to Sam Sanders for further comments on an earlier draft of this work.

A Direct proof of Corollary 3.6

$\mathrm{DC}_\rho \to \mathrm{TP}_\rho[\rhd]$: This uses the famous minimal bad-sequence construction, which in turn uses dependent choice in the following sequential variant:

$$\text{DC-seq}_\rho : \; A([]) \wedge \forall a^{\rho^*}(A(a) \to \exists x^\rho A(a * x)) \to \exists \alpha^{\mathtt{Nat}\to\rho} \forall n A([\alpha](n))$$

For a proof that DC-seq$_\rho$ is implied by DC_{ρ^*} (which can in turn be coded as an instance of DC_ρ) see [17, Section 6.1], and also [13, Lemma 5.11] for the two-way equivalence between DC_ρ and a sequential variant closely related to DC-seq$_\rho$. Let us call a sequence α *bad* if $\neg \mathit{WF}[\succ](\alpha)$ holds: in other words, α is an infinite $\succ$-descending chain. Suppose that the premise of $\mathrm{TP}[\rhd]$ holds, and that for contradiction there exists at least one bad sequence. Using DC-seq$_\rho$ together with $\mathrm{TI}[\rhd]$, construct a *minimal* sequence α as follows:

> Assuming we have already constructed $[\alpha_0, \ldots, \alpha_{n-1}]$, choose α_n in such a way that $[\alpha_0, \ldots, \alpha_{n-1}, \alpha_n]$ extends to a bad sequence, but $[\alpha_0, \ldots, \alpha_{n-1}, x]$ does not for any $x \lhd \alpha_n$.

For the empty sequence in the first step this is guaranteed by the initial assumption that at least one bad sequence exists. It is easy to see that α must satisfy $\mathit{MIN}(\alpha)$. However, α itself must also be bad: if on the contrary we would have $\alpha_n \not\succ \alpha_{n+1}$ for some n, then $[\alpha_0, \ldots, \alpha_{n+1}]$ could not extend to a bad sequence, a contradiction.

$\mathrm{RBI}_\rho \to \mathrm{TP}_\rho[\rhd]$: Define

$$S(a) :\equiv (\forall n < |a|, \beta^{\rho^{\mathtt{Nat}}})([a](n) \blacktriangleleft \beta \wedge a_n \rhd \beta_n \to \mathit{WF}(\beta))$$
$$P(a) :\equiv \forall \alpha (a \blacktriangleleft \alpha \to \mathit{WF}(\alpha)).$$

From the premise of TP we derive the three premises of RBI w.r.t P and S. Note that $S([])$ is trivially true, and if $\alpha \in S$ then this is completely equivalent to saying that $\mathit{MIN}(\alpha)$ holds, and hence $\alpha_n \not\succ \alpha_{n+1}$ for some n and thus $P([\alpha](n+2))$ holds.

For the third premise, take some $a \in S$ and assume that $(\forall x)(S(a * x) \to P(a * x))$. We establish $(\forall x) P(a * x)$ via a side induction on $\rhd$, from which we trivially obtain $P(a)$ since for any α with $a \blacktriangleleft \alpha$ we have $a * \alpha_{|a|} \blacktriangleleft \alpha$ and therefore $\mathit{WF}(\alpha)$ follows from $P(a * \alpha_{|a|})$.

Suppose that $(\forall y \lhd x) P(a * y)$ holds. Then to prove $P(a * x)$ it suffices to prove $S(a * x)$. Since we already have $a \in S$, it suffices to check the last point of the sequence i.e.

$$(\forall\beta)(a \blacktriangleleft \beta \wedge x \rhd \beta_{|a|} \to WF(\beta)).$$

But this follows from the side induction hypothesis, setting $y := \beta_{|a|}$, which completes the side induction. Therefore, we can now apply bar induction to obtain $P([])$ which is just $(\forall\alpha)\, WF(\alpha)$.

References

[1] Baader, F., Nipkow, T.: Term Rewriting and All That. Cambridge University Press (1998)

[2] Berger, U.: A computational interpretation of open induction. In: Proceedings of LICS 2004, pp. 326–334. IEEE Computer Society (2004)

[3] Buchholz, W.: Proof-theoretic analysis of termination proofs. Annals of Pure and Applied Logic **75**, 57–65 (1995)

[4] Coquand, T.: Constructive topology and combinatorics. In: Constructivity in Computer Science, *LNCS*, vol. 613, pp. 159–164 (1991)

[5] Dershowitz, N.: Orderings for term rewriting systems. Theoretical Computer Science **17**(3), 279–301 (1982)

[6] Dershowitz, N.: Termination of rewriting. Journal of Symbolic Computation **3**, 69–116 (1987)

[7] Ferreira, M.C.F., Zantema, H.: Well-foundedness of term orderings. In: N. Dershowitz (ed.) Conditional Term Rewriting Systems (CTRS '94), *LNCS*, vol. 968, pp. 106–123 (1995)

[8] Goubault-Larrecq, J.: Well-founded recursive relations. In: Computer Science Logic (CSL'01), *LNCS*, vol. 2142, pp. 484–498 (2001)

[9] Knuth, D.E., Bendix, P.: Simple word problems in universal algebras. In: Automation of Reasoning. Symbolic Computation, pp. 236–297. Springer (1970)

[10] Kohlenbach, U.: Higher order reverse mathematics. In: Reverse Mathematics 2001, *Lecture Notes in Logic*, vol. 21, pp. 281–295. ASL (2005)

[11] Melliès, P.A.: On a duality between Kruskal and Dershowitz theorem. In: Proceedings of ICALP, *LNCS*, vol. 1443, pp. 518–529 (1998)

[12] Moser, G., Powell, T.: On the computational content of termination proofs. In: Proceedings of Computability in Europe (CiE 2015), *LNCS*, vol. 9136, pp. 276–285 (2015)

[13] Powell, T.: On bar recursive interpretations of analysis. Ph.D. thesis, Queen Mary University of London (2013)

[14] Powell, T.: The equivalence of bar recursion and open recursion. Annals of Pure and Applied Logic **165**(11), 1727–1754 (2014)

[15] Powell, T.: A proof theoretic study of abstract termination principles (2019). To appear: Journal of Logic and Computation

[16] Powell, T.: Well quasi-orders and the functional interpretation. In: P. Schuster, M. Seisenberger, A. Weiermann (eds.) Well-Quasi Orders in Computation, Logic, Language and Reasoning, *Trends in Logic*, vol. 53, pp. 221–269. Springer (2020)

[17] Seisenberger, M.: On the constructive content of proofs. Ph.D. thesis, Ludwig Maximilians Universität München (2003)

[18] Sternagel, C.: A mechanized proof of Higman's lemma by open induction. In: P. Schuster, M. Seisenberger, A. Weiermann (eds.) Well-Quasi Orders in Computation, Logic, Language and Reasoning, *Trends in Logic*, vol. 53, pp. 339–350. Springer (2020)

[19] Troelstra, A.S.: Metamathematical Investigation of Intuitionistic Arithmetic and Analysis, *Lecture Notes in Mathematics*, vol. 344. Springer, Berlin (1973)